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THE USE OF A HIGHER ORDER KINEMATIC RELATIONSHIP ON THE ANALYSIS OF CYLINDRICAL COMPOSITE PANELS

THESIS

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THESIS

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List of Symbols

Symbol	
a	Length in x direction
A_{ij}	Extensional Stiffness Terms
A_{mn}, B_{mn}, C_{mn}, E_{mn}, G_{mn}	Constant coefficients in Admissible Functions
b	Length in y direction
B_{ij}	Coupling Stiffness Terms
D_{ij}	Bending Stiffness Terms
E_i	Young's Modulus in the ith direction
E_{ij}, F_{ij}, G_{ij}, H_{ij}, I_{ij}, J_{ij}	Higher Order Stiffness Terms
ε_i	Normal Strain in the ith direction
G_{ij}	Shear Modulus in the i-j plane
γ_{ij}	Shear Strain in the i-j plane
h	Laminate thickness
I_i, $\bar{I}_i$, $\bar{I}_i'$	Mass Moments of Inertia
k	Constant = $-4/3h^2$
κ_i^j	Midsurface Curvature
L_i, P_i, R_i, S_i	Higher Order Resultant Quantities
m,n	Summation Indices for Galerkin Equations
M,N	Summation Limits for Galerkin Equations
M_i	Moment Resultants
N_i, Q_i	Force Resultants
$\bar{N}_1$, $\bar{N}_2$, $\bar{N}_6$	In-plane Buckling Loads in x, y, and x-y directions
ν_{ij}	Poisson's Ratio
ϕ_{mn}	Generalized Admissible Function

p,q	Summation Indices for Galerkin Equations
Ψ_x	Midsurface Degree of Freedom: Rotation of cross section about y axis
Ψ_y	Midsurface Degree of Freedom: Rotation of cross section about x axis.
Q_{ij}	Reduced Stiffness Terms
$\bar{Q}_{ij}$	Transformed Reduced Stiffness Terms
θ_k	Fiber Orientation of kth ply
R	Radius of Curvature
ρ	Mass Density
S_{ij}	Compliance Terms
σ_x, σ_y	Normal Stress in x and y directions
t_k	Ply thickness
t	Time
T	Kinetic Energy
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shear Stress in x-y, x-z, y-z planes
u_o	Midsurface Displacement in x direction
u	Displacement in x direction
U	Strain Energy
v_o	Midsurface Displacement in y direction
v	Displacement in y direction
V	Potential Energy
w_o	Midsurface Displacement in z direction
w	Displacement in z direction
ω	Frequency of Vibration
ξ	Through the thickness Shear Strain ε_z
ζ	General Degree of Freedom

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Abstract

epsilon sub z

An analytical study was performed to determine the critical buckling loads and natural frequencies for composite cylindrical shells, including transverse shear effects and constant through the thickness direct strain ε_z. A linearized form of Sanders shell equations are derived, including a parabolic transverse shear strain distribution. Higher order laminate constitutive relations are developed. Hamilton's Principle is applied to derive five partial differential equations of motion and the associated boundary conditions, which are then solved using the Galerkin technique.

Ply layups of [0/90], [45/-45], and [0/45/-45/90] were investigated under three boundary conditions, simply supported, clamped, and a combination simple-clamped. Symmetric and nonsymmetric laminates were investigated.

Curvature is shown to have a important effect on all panels investigated, due to membrane and bending coupling. Buckling loads for deeper shells are significantly higher than for flat plates. The effect on frequencies is not as great.

Comparisons between various ply layups and boundaries show results are greatly dependent on the shell geometry, curvature, and boundary conditions.

Behavior of the nonsymmetric laminates is not as expected. Most results indicate the nonsymmetric laminates to be

stiffer than the corresponding symmetric layup, contrary to theoretical predictions.

THE USE OF A HIGHER ORDER KINEMATIC RELATIONSHIP ON THE ANALYSIS OF CYLINDRICAL COMPOSITE PANELS

I. Introduction

Background

Much work has been done in the area of composite cylindrical shells over the past few years. The high strength to weight ratios and the ability to tailor material to meet specific design goals make composite structures ideal for many applications, especially in the aerospace field. Noor and Burton have gathered extensive information on developments in this field (13; 14). A few key developments are presented here.

It has long been known that Classical Plate Theory, based on the Kirchhoff-Love hypothesis, tends to produce large errors when dealing with advanced composite materials. The Kirchhoff-Love hypothesis assumes that straight lines normal to the undeformed shell midsurface remain straight and normal. In other words, transverse shear strains are neglected, resulting in overestimates for the buckling loads and natural frequencies.

Reissner was the first to recognize the need to include the effects of transverse shear effects (23). Mindlin followed, and added the effects of rotary inertia (12). The so called Reissner-Mindlin theory assumes that while cross sections remain plane, they are allowed to rotate around the normal. This theory does not, however, satisfy the boundary conditions of zero transverse shear on the top and bottom surfaces of the shell. This requires application of a correction factor, and is commonly accepted.

Reddy (20; 21; 22) assumed that the displacements of the middle surfaces are cubic functions of z. This leads to a parabolic distribution for the transverse shear strain, and does not require a correction factor. Linneman and Palazotto (10; 16) used this approach in developing solutions for the critical buckling loads and natural frequencies for symmetric laminated cylindrical shells.

Recently, there has been interest focused on nonsymmetric laminates. Nonsymmetric laminates have coupling present between their extensional and bending behavior, and thus are not desirable in many engineering applications. However, sometimes their use cannot be avoided, and delamination in symmetric laminates create a need to understand the effects of nonsymmetry. Whitney (28) applied Donnell-type equations to laminated cylindrical shells. Reddy (21) used a Navier solution for cross-ply antisymmetric shells. Chen and Shu (5)

applied large deformation theory including shear deformation to thick nonsymmetric laminates. Burt and Kumar (1) looked at shells of bimodulus composite materials, which display bending-extension coupling analogous to nonsymmetric laminates.

Objectives

This thesis will apply a higher order shear deformation theory to nonsymmetric composite cylindrical shell panels. Solutions for the critical buckling loads and natural frequencies for different geometries and boundary conditions will be found. Comparisons will be made against solutions for symmetric laminates using similar theories. The effects of varying the radius of curvature and span are also investigated.

Approach

The approach taken for this thesis parallels that used by previous analyses in this area (2; 10; 18). Displacement fields are assumed based on Reddy's parabolic shear strain model (20; 21; 22). Strain displacement relations are developed using a linearized form of Sander's equations (4; 10). Equations of motion are derived using Hamilton's principle (11; 25), and incorporating laminate constitutive relationships (9). Galerkin's Technique is than applied (11; 27). Appropriate admissible functions are chosen to represent each boundary condition studied. The resulting Galerkin equations are then placed into a FORTRAN code that generates an eigenvalue problem. The solutions are the critical buckling load or

the natural frequencies for the particular ply characteristics and geometry specified.

Convergence characteristics are investigated to check the validity of the Galerkin solutions. Comparisons of results are made for boundary conditions, ply orientations, and symmetry.

II. *Theory*

Strain-Displacement Relations

The coordinate system for the circular cylindrical shell

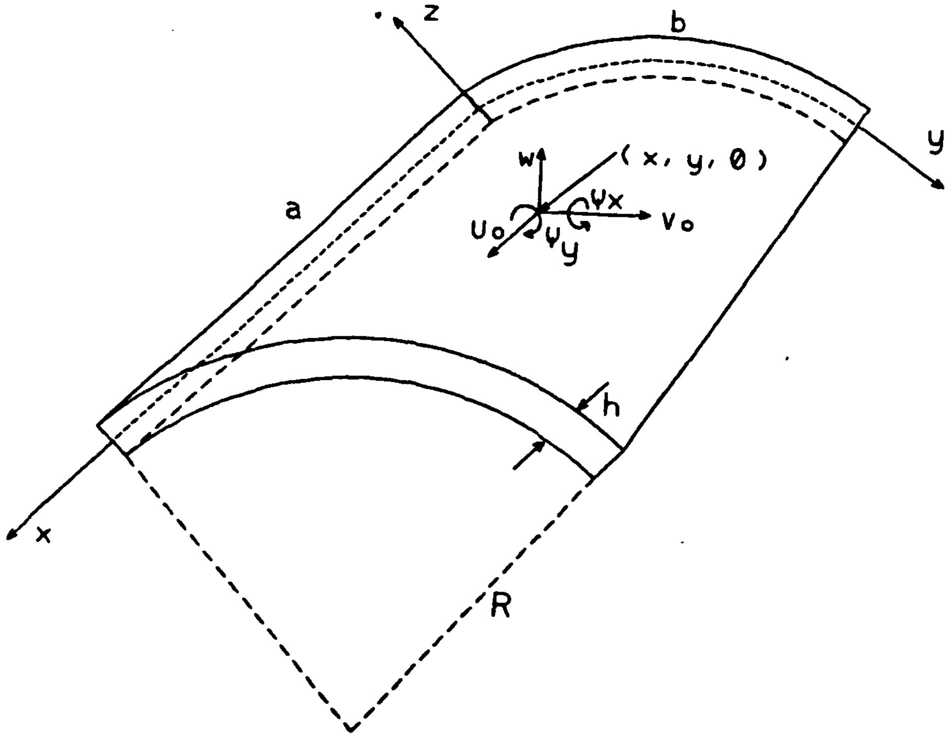

Figure 1. Shell Panel Coordinates

panel is shown in Figure 1. The x and y axes are located at the midsurface of the laminate, at z = 0. The degrees of freedom $u_o(x,y,t)$, $v_o(x,y,t)$, and $w_o(x,y,t)$ are the displace-

ments of the laminate midsurface in the x, y and z directions respectively. The degrees of freedom $\Psi_x(x,y,t)$ and $\Psi_y(x,y,t)$ are the bending rotations of the cross section from the normal with respect to the y and x axes, respectively. R is the radius of curvature, h the laminate thickness, a the length in the x direction, and b the length in the y direction.

The displacement field to be used for this work was presented by Reddy (20; 21; 22), with one modification. The forms for u and v were chosen in order to provide a parabolic shear strain distribution across the thickness of the laminate. In addition, the assumed displacement w is a linear function of z. Whitney (29:315-319) applied this concept to a flat plate. As a result, ε_z is not equal to zero, but is constant across the thickness. Related work has considered ε_z to be zero, thus simplifying the problem (10; 18). The following development will include ε_z as a constant value through the thickness.

The assumed displacement fields, to meet the ideas stated above, can be written as follows:

$$
\begin{aligned}
u(x,y,z,t) &= u_o + z\psi_x + z^2\phi_1 + z^3\theta_1 \\
v(x,y,z,t) &= \left(1 + \frac{z}{R}\right) v_o + z\psi_y + z^2\phi_2 + z^3\theta_2 \\
w(x,y,z,t) &= w_o + z\xi
\end{aligned}
\qquad (1)
$$

(each of the functions shown, u_o, v_o, Ψ_x and Ψ_y are functions of x and y only, not of z). The values of ϕ_1, ϕ_2, θ_1, and θ_2

will be found such that the boundary conditions of zero transverse shear strain at the top and bottom surfaces of the laminate are satisfied. The function $\xi(x,y,t)$ represents the strain ε_z, constant through the thickness.

The strain-displacement relations, for a circular cylindrical shell, are based on Sander's equations (4:195; 7; 24:595). They can be stated as follows:

$$
\begin{aligned}
\varepsilon_x &= u,_x \\
\varepsilon_y &= \frac{1}{1+\dfrac{z}{R}}\left(v,_y + \frac{w}{R}\right) \\
\varepsilon_z &= w,_z \\
\gamma_{xy} &= \frac{1}{1+\dfrac{z}{R}} u,_y + v,_x \\
\gamma_{yz} &= \frac{1}{1+\dfrac{z}{R}}\left(w,_y - \frac{v}{R}\right) + v,_z \\
\gamma_{xz} &= u,_z + w,_x
\end{aligned}
\tag{2}
$$

where $(\ ),_x$ represents partial differentiation with respect to x, etc.

The quantity z/R is assumed to be approximately zero for the transverse shear strains in the yz and xz planes. For the remaining strains, the following binomial expansion is made:

$$\frac{1}{1+\dfrac{z}{R}} = 1 - \frac{z}{R} + \left(\frac{z}{R}\right)^2 - \left(\frac{z}{R}\right)^3 + \ldots\ldots \approx 1 - \frac{z}{R}$$

This approximation allows the strain-displacement relations to hold for deeply curved panels, with a height to radius of curvature ratio up to 1/5 (7).

As shown in Appendix A, if the transverse shear strains, γ_{yz} and γ_{xz}, are set equal to zero at the top and bottom surfaces, the values of the unknown coefficients in the displacement relations are found to be:

$$\phi_1 = \frac{-\xi,_x}{2}, \qquad \phi_2 = \frac{-\xi,_y}{2}$$

$$\theta_1 = k(\psi_x + w,_x), \qquad \theta_2 = k(\psi_y + w,_y)$$

where $k = -4/3h^2$.

The displacement field thus becomes:

$$\begin{aligned} u(x,y,z,t) &= u_o + z\psi_x - z^2\frac{\xi,_x}{2} + z^3k(\psi_x + w_{o,x}) \\ v(x,y,z,t) &= \left(1 + \frac{z}{R}\right)v_o + z\psi_y - z^2\frac{\xi,_y}{2} + z^3k(\psi_y + w,_y) \\ w(x,y,z,t) &= w_o + z\xi \end{aligned} \tag{6}$$

Using these displacements in Sander's equations, Eq (2), the strain-displacement equations become:

$$\varepsilon_x = u_{o,x} + z\psi_{x,x} - z^2\frac{\xi_{,xx}}{2} + z^3k(\psi_{x},x + w_{o,xx})$$

$$\varepsilon_y = v_{o,y} + \frac{w_o}{R} + z\left(\psi_{y,y} + \frac{\xi}{R}\right) - z^2\left(\frac{\psi_{y,y}}{R} + \frac{\xi_{,yy}}{2}\right)$$

$$+ z^3\left[k(\psi_{y,y} + w_{o,yy}) + \frac{\xi_{,yy}}{2R}\right] - z^4\frac{k}{R}(\psi_{y,y} + w_{o,yy})$$

$$\varepsilon_z = \xi$$

$$\gamma_{xy} = u_{o,y} + v_{o,x} + z\left(\psi_{x,y} + \psi_{y,x} + \frac{1}{2R}(v_{o,x} - u_{o,y})\right) - z^2\left(\frac{\psi_{x,y}}{R} + \xi_{,xy}\right)$$

$$+ z^3k(\psi_{x,y} + \psi_{y,x} + 2w_{o,xy}) - z^4\frac{k}{R}(\psi_{x,y} + w_{o,xy})$$

$$\gamma_{yz} = \psi_y + w_{o,y} + 3z^2k(\psi_y + w_{o,y})$$

$$\gamma_{xz} = \psi_x + w_{o,x} + 3z^2k(\psi_x + w_{o,x}) \qquad \textbf{(7)}$$

These expressions can be presented in matrix notation to simplify later usage,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_z^o \\ \gamma_{xy}^o \\ \gamma_{yz}^o \\ \gamma_{xz}^o \end{Bmatrix} + z\begin{Bmatrix} \kappa_x^o \\ \kappa_y^o \\ 0 \\ \kappa_{xy}^o \\ 0 \\ 0 \end{Bmatrix} + z^2\begin{Bmatrix} \kappa_x^1 \\ \kappa_y^1 \\ 0 \\ \kappa_{xy}^1 \\ \kappa_{yz}^1 \\ \kappa_{xz}^1 \end{Bmatrix} + z^3\begin{Bmatrix} \kappa_x^2 \\ \kappa_y^2 \\ 0 \\ \kappa_{xy}^2 \\ 0 \\ 0 \end{Bmatrix} + z^4\begin{Bmatrix} 0 \\ \kappa_y^3 \\ 0 \\ \kappa_{xy}^3 \\ 0 \\ 0 \end{Bmatrix} \qquad \textbf{(8)}$$

Note the superscripts on the κ_i terms are not exponents, but for identification purposes only, to distinguish among the

higher order terms (21;22). The strains at the midsurface and curvature terms are defined below:

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_z^o \\ \gamma_{xy}^o \\ \gamma_{yz}^o \\ \gamma_{xz}^o \end{Bmatrix} = \begin{Bmatrix} u_{o,x} \\ v_{o,y} + \dfrac{w_o}{R} \\ \xi \\ u_{0,y} + v_{o,x} \\ \psi_y + w_{o,y} \\ \psi_x + w_{o,x} \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} \kappa_x^0 \\ \kappa_y^o \\ \kappa_{xy}^o \end{Bmatrix} = \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} + \dfrac{\xi}{R} \\ \psi_{x,y} + \psi_{y,x} + \dfrac{1}{2R}(v_{o,x} - u_{o,y}) \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \kappa_x^1 \\ \kappa_y^1 \\ \kappa_{xy}^1 \\ \kappa_{yz}^1 \\ \kappa_{xz}^1 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\xi_{,xx}}{2} \\ -\dfrac{\psi_{y,y}}{R} - \dfrac{\xi_{,yy}}{2} \\ -\dfrac{\psi_{x,y}}{R} - \xi_{,xy} \\ 3k(\psi_y + w_{o,y}) \\ 3k(\psi_x + w_{o,x}) \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \kappa_x^2 \\ \kappa_y^2 \\ \kappa_{xy}^2 \end{Bmatrix} = \begin{Bmatrix} k(\psi_{x,x} + w_{o,xx}) \\ k(\psi_{y,y} + w_{o,yy}) + \dfrac{\xi_{,yy}}{2R} \\ k(\psi_{x,y} + \psi_{y,x} + 2w_{o,xy}) \end{Bmatrix} \tag{12}$$

$$\begin{Bmatrix} \kappa_y^3 \\ \kappa_{xy}^3 \end{Bmatrix} = \begin{Bmatrix} -\frac{k}{R}\left(\psi_{y,y} + w_{o,yy}\right) \\ -\frac{k}{R}\left(\psi_{x,y} + w_{o,xy}\right) \end{Bmatrix} \tag{13}$$

Equations of Motion

The displacements and strain relations will now be used to find the equations of motion and associated boundary conditions. The basis for this development is Hamilton's Principle, which states

$$\int_{t_1}^{t_2} \delta\, (T - U - V)\, dt = 0 \tag{14}$$

where T is the kinetic energy, U is the strain energy, and V is the potential energy due to external forces (11; 19; 25). The symbol δ represents the first variation of the enclosed quantities.

The kinetic energy is defined as

$$T = \int_0^b \int_0^a \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}\rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) dz\, dx\, dy \tag{15}$$

where ρ is the mass density.

It was decided to focus the scope of this thesis on investigating nonsymmetric ply lay-ups. Therefore, in order to isolate the effect of the nonsymmetry from the transverse

shear effects, the direct strain ε_z was eliminated from the strain relationships by assuming the value of ξ to be zero. The displacements and strain relations thus become those presented by Linneman (10:12).

Based on these simplified equations, the squares of the partial time derivatives of the displacements are as follows:

$$\dot{u}^2 = \dot{u}_o^2 + (2z + 2kz^3)\,\dot{u}_o\dot{\psi}_x + 2kz^3\dot{u}_o\dot{w},_x + (z^2 + 2kz^4 + k^2z^6)\,\dot{\psi}_x^2$$
$$+ (2kz^4 + 2kz^6)\,\dot{\psi}_x\dot{w},_x$$
$$\dot{v}^2 = \left(1 + 2\frac{z}{R}\right)\dot{v}_o^2 + \left(1 + \frac{z}{R}\right)(2z + 2kz^3)\,\dot{v}_o\dot{\psi}_y + 2\left(1 + \frac{z}{R}\right)kz^3\dot{v}_o\dot{w},_y$$
$$+ (z^2 + 2kz^4 + k^2z^6)\,\dot{\psi}_y^2 + (2kz^4 + 2k^2z^6)\,\dot{\psi}_y\dot{w},_y + k^2z^6\dot{w},_y^2$$
$$\dot{w}^2 = \dot{w}^2 \tag{16}$$

At this time, the following definitions for the mass moments of inertia are introduced:

$$(\, I_1, I_2, I_3, I_4, I_5, I_7 \,) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\,(1, z, z^2, z^3, z^4, z^6)\; dz \tag{17}$$

with the following simplifications (21; 22):

$$\overline{I}_1' = I_1 + \frac{2}{R}I_2$$
$$\overline{I}_2 = I_2 + kI_4$$

$$\overline{I}_2' = I_2 + \frac{1}{R}I_3 + kI_4 + \frac{k}{R}I_5$$

$$\overline{I}_3 = -k\overline{I}_4$$

$$\overline{I}_3' = -kI_4 - \frac{k}{R}I_5$$

$$\overline{I}_4 = I_3 + 2kI_5 + k^2 I_7$$

$$\overline{I}_5 = -kI_5 - k^2 I_7$$

The kinetic energy becomes:

$$T = \frac{1}{2}\int_0^b\int_0^a (I_1\dot{u}_o^2 + 2\overline{I}_2\dot{u}_o\dot{\psi}_x - 2\overline{I}_3\dot{u}_o\dot{w}_{,x} + \overline{I}_4\dot{\psi}_x^2 - 2\overline{I}_5\dot{\psi}_x\dot{w}_{,x}$$
$$+ k^2 I_7\dot{w}_{,x}^2 + \overline{I}_1'\dot{v}_o^2 + 2\overline{I}_2'\dot{v}_o\dot{\psi}_y - 2\overline{I}_3'\dot{v}_o\dot{w}_{,y} + \overline{I}_4\dot{\psi}_y^2$$
$$- 2\overline{I}_5\dot{\psi}_y\dot{w}_{,y}^2 + I_1\dot{w}^2)\ dx\ dy$$

(19)

Taking the first variation

$$\delta T = \int_0^b\int_0^a [(I_1\dot{u}_o + \overline{I}_2\dot{\psi}_x - \overline{I}_3\dot{w}_{,y})\delta\dot{u}_o + (\overline{I}_2\dot{u}_o + \overline{I}_4\dot{\psi}_x - \overline{I}_5\dot{w}_{,x})\delta\dot{\psi}_x$$
$$+ (-\overline{I}_3\dot{u}_o - \overline{I}_5\dot{\psi}_x + k^2 I_7\dot{w}_{,x})\delta\dot{w}_x + (\overline{I}_1'\dot{v}_o + \overline{I}_2\dot{\psi}_y - \overline{I}_3'\dot{w}_{,y})\delta\dot{v}_o$$
$$+ (\overline{I}_2'\dot{v}_o + \overline{I}_4\dot{\psi}_y - \overline{I}_5\dot{w}_{,y})\delta\dot{\psi}_y + (\overline{I}_3'\dot{v}_o - \overline{I}_5\dot{\psi}_y + k^2 I_7\dot{w}_{,y})\delta\dot{w}_y$$
$$+ I_1\dot{w}\delta\dot{w}]\ dx\ dy$$

(20)

The final form of the variation in kinetic energy is obtained by integrating by parts the quantities $\delta\dot{w}_{,x}$ and $\delta\dot{w}_{,y}$ with respect to x and y, and then integrating the entire expression with respect to time. As described by Meirovitch (11:45-46), the variations of the degrees of freedom over the interval t_1 and t_2 are zero; thus this contribution to the integration by parts vanishes. This leaves the following:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2}\int_0^b\int_0^a \Big[-(I_1\ddot{u}_o + \overline{I}_2\ddot{\psi}_x - \overline{I}_3\ddot{w}_{,x})\delta u_o - (\overline{I}_1'\ddot{v}_o + \overline{I}_2'\ddot{\psi}_y - \overline{I}_3'\ddot{w}_{,y})\delta v_o + [\overline{I}_3\ddot{u}_{0,x} + \overline{I}_5\ddot{\psi}_{x,x} + \overline{I}_3'\ddot{v}_{o,y} - k^2 I_7(\ddot{w}_{,xx} + \ddot{w}_{yy}) + \overline{I}_5\ddot{\psi}_{y,y} + I_5\ddot{w}]\delta w - (\overline{I}_2\ddot{u}_o + \overline{I}_4\ddot{\psi}_x - \overline{I}_5\ddot{w}_{,x})\delta\psi_x - (\overline{I}_2'\ddot{v}_o + \overline{I}_4\ddot{\psi}_y - \overline{I}_5\ddot{w}_{,y})\delta\psi_y \Big)\, dxdydt \qquad \textbf{(21)}$$

The strain energy is developed according to procedures presented in (11), (19), and (25). The first variation of the strain is written as follows:

$$\delta U = \int_0^b\int_0^a\int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})\, dxdydz \qquad \textbf{(22)}$$

It is convenient at this point in the development to introduce the constitutive stress-strain relationships for the

laminated structure, as these relations will result in force resultants that will simplify the strain energy formulations.

As defined by Jones (9:45-51), the constitutive relations for a single orthotropic layer in the principle coordinate system shown in Figure 2 are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$
$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \tag{23}$$

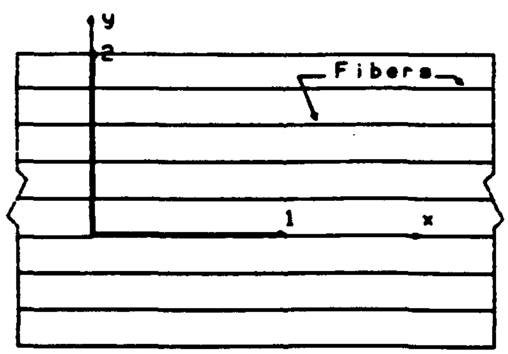

Figure 2. Lamina Material Coordinates

The S_{ij} are compliance terms. They can be written in terms of the engineering constants as follows:

$$
\begin{aligned}
S_{11} &= \frac{1}{E_{11}} \qquad & S_{12} &= -\frac{\nu_{21}}{E_2} \\
S_{22} &= \frac{1}{E_2} \qquad & S_{66} &= \frac{1}{G_{12}} \\
S_{44} &= \frac{1}{G_{23}} \qquad & S_{55} &= \frac{1}{G_{13}}
\end{aligned}
\tag{24}
$$

Here E_i are the Young's moduli in the ith direction, ν_{ij} are the Poisson's ratios, and G_{ij} are the shear moduli in the corresponding i-j plane.

The above equation can be inverted to obtain the relationship for stress in terms of strain:

$$
\begin{gathered}
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \\
\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}
\end{gathered}
\tag{25}
$$

Here, Q_{ij} are the reduced stiffness terms and are defined as

$$
\begin{gathered}
Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\
Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{31}
\end{gathered}
\tag{26}
$$

Eq (26) applies only to a laminate in which the fiber orientation (1-2 coordinates) coincides with the x-y coordinate axes of the structure. Generally, this is not the case. For laminae with the fibers oriented at some angle θ from the x-y axes as shown in Figure 3 below, the reduced stiffness matrix $[Q_{ij}]$ of Eq (26) must be transformed to reflect the rotation of the laminate axes. The transformation matrices are defined below, where m = cos θ and n = sin θ

$$[Q_{ij}] \quad i,j = 1,2,6 \quad use \quad T = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \tag{27}$$

$$[Q_{ij}] \quad i,\ j = 4,\ 5 \quad use \quad T = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \tag{28}$$

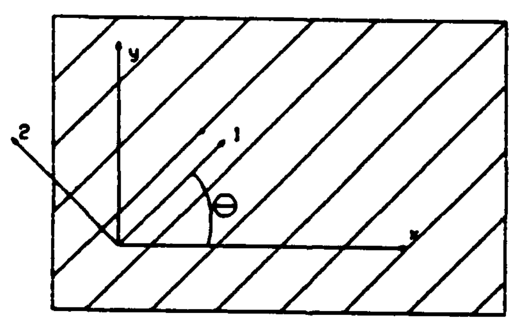

Figure 3. Arbitrary Lamina Coordinates

The transformed stiffness matrices are thus found

$$[\bar{Q}_{ij}] = [T]^{-1}\,[Q_{ij}]\,[T]^{-T} \tag{29}$$

The lamina constitutive relations can now be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{30}$$

where k denotes the kth lamina. The $\bar{Q}_{ij}$ terms are given by the following simplified relationships:

$$\bar{Q}_{11} = Q_{11}\cos^4\theta = 2(Q_{12} + Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\cos^4\theta + \sin^4\theta)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$

$$\bar{Q}_{45} = (Q_{44} - Q_{55})\cos\theta\sin\theta$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \tag{31}$$

The final form for the stress in the kth lamina is found by substituting the strain expressions and the transformed reduced stiffness terms into Eq (31)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x^o \\ \kappa_y^o \\ \kappa_{xy}^o \end{Bmatrix} + z^2 \begin{Bmatrix} 0 \\ \kappa_y^1 \\ \kappa_{xy}^1 \end{Bmatrix} + z^3 \begin{Bmatrix} \kappa_x^2 \\ \kappa_y^2 \\ \kappa_{xy}^2 \end{Bmatrix} + z^4 \begin{Bmatrix} 0 \\ \kappa_y^3 \\ \kappa_{xy}^3 \end{Bmatrix} \right]$$

$$\begin{Bmatrix} \tau_{yx} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \left[\begin{Bmatrix} \gamma_{yz}^o \\ \gamma_{xz}^o \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_{yz}^1 \\ \kappa_{xz}^1 \end{Bmatrix} \right] \tag{32}$$

The stress over the entire laminate is obtained by integrating the individual laminae stresses over the thickness. This produces quantities representing the resultant forces and moments, plus higher order terms, acting over the laminate. These are shown below:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix}, \begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix}, \begin{Bmatrix} S_1 \\ S_2 \\ S_6 \end{Bmatrix}, \begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix}, \begin{Bmatrix} L_1 \\ L_2 \\ L_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} (1, z, z^2, z^3, z^4)\, dz$$

$$= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_x \end{Bmatrix} (1, z, z^2, z^3, z^4)\, dz$$

$$\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix}, \begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} (1, z^2)\, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \tau_{yx} \\ \tau_{xz} \end{Bmatrix}_k (1, z^2)\, dz \tag{33}$$

The quantities $\{N_i\}$, $\{M_i\}$, and $\{Q_i\}$, represent the resultant forces and moments found in conventional laminated plate theories (9:154; 29:239). The remaining quantities $\{S_i\}$, $\{P_i\}$, $\{L_i\}$, and $\{R_i\}$ are higher order resultants from the parabolic distribution of the transverse shear strain (10; 21).

The expressions for the stresses, in terms of the transformed reduced stiffness matrices, Eq (32), and the strain relations, Eq (8), are now substituted into the above expressions, and the terms independent of z brought outside of the integral. Finally, the following notation for the laminate stiffness matrices is introduced:

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},G_{ij},H_{ij},I_{ij},J_{ij}) = \sum_{k=1}^{N} \left[\overline{Q}_{ij}\right]_k \int_{z_{k-1}}^{z_k} (1,z,z^2,z^3,z^4,z^5,z^6,z^7,z^8)\,dz \qquad i = 1,2,6$$

$$(A_{ij},D_{ij},F_{ij}) = \sum_{k=1}^{N} \left[\overline{Q}_{ij}\right]_k \int_{z_{k-1}}^{z_k} (1,z^2,z^4)\,dz \quad i = 4,5 \qquad \textbf{(34)}$$

Eq (33) may now be written in matrix expanded form:

$$
\begin{matrix}
\begin{Bmatrix} N_1 \\ N_2 \\ N_6 \end{Bmatrix} \\
\begin{Bmatrix} M_1 \\ M_2 \\ M_6 \end{Bmatrix} \\
\begin{Bmatrix} S_1 \\ S_2 \\ S_6 \end{Bmatrix} \\
\begin{Bmatrix} P_1 \\ P_2 \\ P_6 \end{Bmatrix} \\
\begin{Bmatrix} L_1 \\ L_2 \\ L_6 \end{Bmatrix}
\end{matrix}
=
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{16} & D_{12} & D_{22} & D_{26} & E_{12} & E_{22} & E_{16} & F_{12} & F_{22} & F_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} \\
 & & & D_{11} & D_{12} & D_{16} & E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & G_{11} & G_{12} & G_{66} \\
 & & & D_{12} & D_{22} & D_{26} & E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & G_{12} & G_{22} & G_{26} \\
 & & & D_{16} & D_{26} & D_{66} & E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & G_{16} & G_{26} & G_{66} \\
 & & & & & & F_{11} & F_{12} & F_{16} & G_{11} & G_{12} & G_{16} & H_{11} & H_{12} & H_{16} \\
 & & & & & & F_{12} & F_{22} & F_{26} & G_{12} & G_{22} & G_{26} & H_{12} & H_{22} & H_{26} \\
 & & & & & & F_{16} & F_{26} & F_{66} & G_{16} & G_{26} & G_{66} & H_{16} & H_{26} & H_{66} \\
 & & S & Y & M & M & & & & H_{11} & H_{12} & H_{16} & I_{11} & I_{12} & I_{16} \\
 & & & & & & & & & H_{12} & H_{22} & H_{26} & I_{12} & I_{22} & I_{26} \\
 & & & & & & & & & H_{16} & H_{26} & H_{66} & I_{16} & I_{26} & I_{66} \\
 & & & & & & & & & & & & J_{11} & J_{12} & J_{16} \\
 & & & & & & & & & & & & J_{12} & J_{22} & J_{26} \\
 & & & & & & & & & & & & J_{16} & J_{26} & J_{66}
\end{bmatrix}
\begin{matrix}
\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_{xy}^o \end{Bmatrix} \\
\begin{Bmatrix} \kappa_x^o \\ \kappa_y^o \\ \kappa_{xy}^o \end{Bmatrix} \\
\begin{Bmatrix} 0 \\ \kappa_y^1 \\ \kappa_{xy}^1 \end{Bmatrix} \\
\begin{Bmatrix} \kappa_x^2 \\ \kappa_y^2 \\ \kappa_{xy}^2 \end{Bmatrix} \\
\begin{Bmatrix} 0 \\ \kappa_y^3 \\ \kappa_{xy}^3 \end{Bmatrix}
\end{matrix}
$$

$$
\begin{matrix}
\begin{Bmatrix} Q_2 \\ Q_1 \end{Bmatrix} \\
\begin{Bmatrix} R_2 \\ R_1 \end{Bmatrix}
\end{matrix}
=
\begin{bmatrix}
A_{44} & A_{45} & D_{44} & D_{45} \\
A_{45} & A_{55} & D_{45} & D_{55} \\
D_{44} & D_{45} & F_{44} & F_{45} \\
D_{45} & D_{55} & F_{45} & F_{55}
\end{bmatrix}
\begin{matrix}
\begin{Bmatrix} \gamma_{yz}^o \\ \gamma_{xz}^o \end{Bmatrix} \\
\begin{Bmatrix} \kappa_{yz}^1 \\ \kappa_{xz}^1 \end{Bmatrix}
\end{matrix}
\qquad (35)
$$

With these expressions now in hand, the formulation of the remaining terms in the energy development will be much simpler. Substituting the strain-displacement relations into

the variation of the strain energy and integrating with respect to z, the resultant quantities from Eq (35) may be substituted to produce the following:

$$\delta U = \int_0^b \int_0^a (N_1 \delta \varepsilon_x^o + M_1 \delta \kappa_x^o + P_1 \delta \kappa_x^o + N_2 \delta \varepsilon_y^o + M_2 \delta \kappa_y^o + S_2 \delta \kappa_y^1$$
$$+ P_2 \delta \kappa_y^2 + L_2 \delta \kappa_y^3 + N_6 \delta \gamma_{xy}^o + M_6 \delta \kappa_{xy}^o + S_6 \delta \kappa_{xy}^1 + P_6 \delta \kappa_{xy}^2$$
$$+ L_6 \delta \kappa_{xy}^3 + Q_2 \delta \gamma_{yz}^o + R_2 \delta \kappa_{yz}^1 + Q_1 \delta \gamma_{xz}^o + R_1 \delta \kappa_{xz}^1)\, dxdy \tag{36}$$

The expressions for the strains at the midsurface and curvature terms, Eq (9) through (13), are now substituted into the above expression. After collecting terms, one obtains

$$\delta U = \int_0^b \int_0^a [N_1 \delta u_{o,x} + (N_6 \ \frac{1}{2R} M_6) \delta u_{o,y} + N_2 \delta v_{o,y} + \frac{1}{R} N_2 \delta w$$
$$+ (N_6 + \frac{1}{2R} M_6) \delta v_{o,x} + k P_1 \delta w_{,xx} + (k P_2 - \frac{1}{R} k L_2) \delta w_{,yy}$$
$$+ (2 k P_6 - \frac{1}{R} k L_6) \delta w_{,xy} + (Q_2 + 3 k R_2) \delta w_{,y} + (Q_1 + 3 k R_1) \delta w_{,x}$$
$$+ (M_1 + k P_1) \delta \psi_{x,x} + (Q_1 + 3 k R_1) \delta \psi_x + (Q_2 + 3 k R_2) \delta \psi_y$$
$$+ (M_2 - \frac{1}{R} S_2 + k P_2 - \frac{1}{R} k L_2) \delta \psi_{y,y} + (M_6 + k P_6) \delta \psi_{y,x}$$
$$+ (M_6 - \frac{1}{R} S_6 + k P_6 - \frac{1}{R} k L_6) \delta \psi_{x,y}]\ dxdy \tag{37}$$

After integrating by parts, the final expression for the variation of the strain energy is obtained:

$$\delta U = \int_0^b \int_0^a [(-N_{1,x} - N_{6,y} + \frac{1}{2R}M_{6,y})\delta u_o + (-N_{2,y} - N_{6,x} - \frac{1}{2R}M_{6,x})\delta v_o$$

$$+ [k(P_{1,xx} + P_{2,yy} + 2P_{6,xy} - Q_{2,y} - Q_{1,x} - 3k(R_{2,y} + R_{1,x})$$

$$+ \frac{1}{R}(N_2 - k(L_{2,yy} + L_{6,xy})]\delta w + [3kR_1 - k(P_{1,x} + P_{6,y}) - M_{1,x}$$

$$- M_{6,y} + Q_1 + \frac{1}{R}(S_{6,y} + kL_{6,y})]\delta\psi_x + [3kR_2 - k(P_{2,y} + P_{6,x})$$

$$- M_{2,y} + \frac{1}{R}(S_{2,y} + kL_{2,y}) - M_{6,x} + Q_2]\delta\psi_y]\, dxdy$$

$$+ \int_0^b [N_1\delta u_o + (N_6 + \frac{1}{2R}M_6)\delta v_o + [-k(P_{1,x} + 2P_{6,y}) + Q_1 + 3kR_1$$

$$+ \frac{1}{R}kL_{6,y}]\delta w + (M_1\ 2kP_1)\delta\psi_x + (M_6 + kP_6)\, dleta\psi_y]\Big|_{x=0}^{x=a} dy$$

$$+ \int_0^a [(N_6 - \frac{1}{2R}M_6)\delta u_o + N_2\delta v_o + [-k(P_{2,y} + 2P_{6,x}) + Q_2 + 3kR_2$$

$$+ \frac{1}{R}k(L_{2,y} + L_{6,x})]\delta w + [M_6 + kP_6 + \frac{1}{R}(-S_6 - kL_6)]\delta\psi_x + [M_2$$

$$+ 2kP_2 + \frac{1}{R}(-S_2 - 2kL_2)]\delta\psi_y]\Big|_{y=0}^{y=b} dx$$

$$+ k[2P_6 - \frac{1}{R}L_6]\delta w\Big|_{y=0}^{y=b}\Big|_{x=0}^{x=a} \tag{38}$$

The final component in Hamilton's Principle is the potential energy, V, of in-plane forces. It is defined as

$$V = \int_0^b \int_0^a (\overline{N}_1\varepsilon_x + \overline{N}_2\varepsilon_y + \overline{N}_6\gamma_{xy})dxdy \tag{39}$$

where $\overline{N}_1$, $\overline{N}_2$, and $\overline{N}_6$, are the initial in-plane loads, and ε_x, ε_y, and γ_{xy} are the midplane strains due to the displacement w.

These strains are normally considered in large deflection analysis, and only nonlinear bending terms, those involving w, are considered. In linear theory, these second order strains are used in determining critical buckling loads (29:241). For the problem presented here, the strains take the following form:

$$\begin{aligned} \varepsilon_x &= \frac{1}{2} w,_x^2 \\ \varepsilon_y &= \frac{1}{2} w,_y^2 + \frac{w}{R} \\ \gamma_{xy} &= w,_x w,_y \end{aligned} \tag{40}$$

The first variation of the potential energy is

$$\delta V = \int_0^b \int_0^a \left[(\bar{N}_1 w,_x + \bar{N}_6 w,_y)\,\delta w,_x + (\bar{N}_2 w,_y + \bar{N}_6 w,_x)\,\delta w,_y + \frac{1}{R}\bar{N}_2 \delta w \right] dxdy \tag{41}$$

After integrating by parts, the final form of the first variation of the potential energy is as follows:

$$\delta V = \int_0^b \int_0^a \left[-\overline{N}_1 w,_{xx} - 2\overline{N}_6 w,_{xy} + \overline{N}_2 \left(\frac{1}{R} - w,_{yy} \right) \right] \delta w dx dy$$

$$+ \int_0^b \left(\overline{N}_1 w,_{x} + \overline{N}_6 w,_{y} \right) \delta w \Big|_0^a dy + \int_0^a \left(\overline{N}_2 w,_{y} + \overline{N}_6 w,_{x} \right) \delta w \Big|_0^b dx \qquad \textbf{(42)}$$

The expressions for the first variations of the kinetic, strain, and potential energies are now substituted back into Hamilton's Principle, Eq (14), to produce the following:

$$\int_{t_1}^{t_2} \int_0^b \int_0^a \Big\{ (-I_1 \ddot{u}_o - \overline{I}_2 \ddot{\psi}_x + \overline{I}_3 \ddot{w},_x + N_{1,x} + N_{6,y} - \frac{1}{2R} M_{7,y}) \delta u_o$$

$$+ (-\overline{I}_1' \ddot{v}_o + \overline{I}_2' \ddot{\psi}_y + \overline{I}_3' \ddot{w},_y + N_{2,y} + N_{6,x} + \frac{1}{2R} M_{6,x}) \delta v_0$$

$$+ [-\overline{I}_3 \ddot{u}_{o,x} - \overline{I}_5 \ddot{\psi}_{x,x} - \overline{I}_3' \ddot{v}_{o,y} + k^2 I_7 (\ddot{w},_{xx} - \ddot{w},_{yy}) - \overline{I}_5 \ddot{\psi}_{y,y}$$

$$- I_1 \ddot{w} - k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} + Q_{1,x} + 3k(R_{2,y}$$

$$+ R_{1,x}) - \frac{1}{R}(N_2 - k(L_{2,yy} + L_{6,xy})) + \overline{N}_1 w,_{xx} + 2\overline{N}_6 w,_{xy}$$

$$- \overline{N}_2 (\frac{1}{R} - w,_{yy})] \delta w \, dx dy dt$$

$$+ [-\overline{I}_2 \ddot{u}_o - \overline{I}_4 \ddot{\psi}_x + \overline{I}_5 \ddot{w},_x + k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y})$$

$$- 3kR_1 - Q_1 - \frac{1}{R}(S_{6,y} + kL_{6,y})] \delta \psi_x + [-\overline{I}_2' \ddot{v}_o - \overline{I}_4 \ddot{\psi}_y$$

$$+ \overline{I}_5 \ddot{w},_y + k(P_{2,y} + P_{6,x}) + M_{2,y} + M_{6,x} - 3kR_2 - Q_2$$

$$- \frac{1}{R}(S_{2,y} + kL_{2,y})] \delta \psi_y \Big\} \, dx dy dt$$

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{0}^{b}\Big\{ N_1\delta u_o + [N_6 + \frac{1}{2R}M_6]\delta v_o + [-k(P_{1,x} + 2P_{6,y} + Q_1 + 3kR_1 \\
&\qquad + \frac{1}{R}kL_{6,y} + \bar{N}_1 w_{,y}]\delta w + (M_1 + 2kP_1)\delta\psi_x + (M_6 + kP_6)\delta\psi_y \Big\}\Big|_{x=0}^{x=a} dydt \\
&-\int_{t_1}^{t_2}\int_{0}^{a}\Big\{ [N_6 - \frac{1}{2R}M_6]\delta u_o + N_2\delta v_o + [-k(P_{2,y} + 2P_{6,x}) + Q_2 + 3kR_2 \\
&\qquad + \frac{1}{R}k(L_{2,y} + L_{6,x}) + \bar{N}_2 w_{,y} + \bar{N}_6 w_{,x}]\delta w + [M_6 + kP_6 \\
&\qquad - \frac{1}{R}(S_6 + kL_6)]\delta\psi_x + [M_2 + 2kP_2 - \frac{1}{R}(S_2 + 2kl_2)]\delta\psi_y \Big\}\Big|_{y=0}^{y=b} dxdt \\
&-\int_{t_1}^{t_2} \Big\{ k[2P_6 - \frac{1}{R}L_6]\delta w \Big\}\Big|_{y=0}^{y=b}\Big|_{x=0}^{x=a} dt = 0
\end{aligned}
\tag{43}
$$

The double integral in the above equation contains the five equations of motion. The two line integrals represent the geometric and natural boundary conditions along the edges of the shell, and the fourth expression gives the boundary conditions at the corners.

The variations of the degrees of freedom δu_o, δv_o, δw, $\delta\Psi_x$, and $\delta\Psi_y$, are arbitrary and generally not equal to zero. Therefore to satisfy Hamilton's Principle, their corresponding coefficients must equal zero. The five coupled partial differential equations of motion for the panel at any time t are therefore defined as follows:

Corresponding with the degree of freedom δu_o:

$$N_{1,x} + N_{6,y} - \frac{1}{2R}M_{6,y} = I_1\ddot{u}_o + \bar{I}_2\ddot{\Psi}_x - \bar{I}_3\ddot{w},_x \tag{44}$$

with δv_o:

$$N_{2,y} + N_{6,x} + \frac{1}{2R}M_{6,y} = \bar{I}_1'\ddot{v}_o + \bar{I}_2'\ddot{\Psi}_y - \bar{I}_3'\ddot{w},_y \tag{45}$$

with δw:

$$\begin{aligned} &-k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} + Q_{1,x} + 3k(R_{2,y} + R_{1,x}) \\ &-\frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})] + \bar{N}_I w,_{xx} + 2\bar{N}_6 w,_{xy} - \bar{N}_2\left[\frac{1}{R} - w,_{yy}\right] \\ &= \bar{I}_3\ddot{u}_{o,x} + \bar{I}_5\ddot{\Psi}_{x,x} + \bar{I}_3'\ddot{v}_{o,y} - k^2 I_7(\ddot{w},_{xx} + \ddot{w},_{yy}) + \bar{I}_5\ddot{\Psi}_{y,y} + I_1\ddot{w} \end{aligned} \tag{46}$$

with $\delta\Psi_x$:

$$\begin{aligned} &k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y} - 3kR_1 - Q_1 - \frac{1}{R}(S_{6,y} + kL_{6,y}) \\ &= \bar{I}_2\ddot{u}_o + \bar{I}_4\ddot{\Psi}_x - \bar{I}_5\ddot{w},_x \end{aligned} \tag{47}$$

with $\delta\Psi_y$:

$$\begin{aligned} &k(P_{2,y} + P_{6,x}) + M_{2,y} + M_{6,x} - 3kR_2 - Q_2 - \frac{1}{R}(S_{2,y} + kL_{2,y}) \\ &= \bar{I}_2'\ddot{v}_o + \bar{I}_4\ddot{\Psi}_y - \bar{I}_5\ddot{w},_y \end{aligned} \tag{48}$$

The equations of motion can be simplified to other forms for certain applications. If R is taken to infinity, they reduce to the equations for a flat plate with parabolic transverse shear and rotary inertia (19; 20; 22). With the following terms neglected:

$$\delta u_o: \quad \frac{1}{2R}M_{6,y}$$

$$\delta v_o: \quad \frac{1}{2R}M_{6,x}$$

$$\delta w: \quad \frac{1}{R}k(L_{2,yy} + L_{6,xy})$$

$$\delta \psi_x: \quad \frac{1}{R}(S_{6,y} + kL_{6,y})$$

$$\delta \psi_y: \quad \frac{1}{R}(S_{2,Y} + kL_{2,y})$$

the equations of motions reduce to Donnell's equations. (21)

The equations developed up to this point have been general in nature. For this thesis, some assumptions have been made, and are introduced at this point.

First, the individual laminae are assumed to have identical material properties. Only the orientation angle θ will change between them. The mass density, ρ is a constant across the thickness. Thus, integrating the inertia terms defined in Eq (17) yields the following:

$$I_2 = I_4 = \overline{I}_2 = \overline{I}_3 = 0$$

$$I_1 = \overline{I}_1' = \rho h$$

$$I_3 = \frac{\rho h^3}{12}, \quad I_5 = \frac{\rho h^5}{80R}, \quad I_7 = \frac{\rho h^7}{448}$$

$$\overline{I}_2' = \frac{\rho h^3}{15R}, \quad \overline{I}_3' = \frac{\rho h^3}{60R}, \quad \overline{I}_4 = \frac{17\rho h^3}{315}, \quad \overline{I}_5 = \frac{4\rho h^3}{315} \tag{50}$$

In addition, in-plane inertia is not considered here, only rotary inertia, as in-plane inertia tends to increase the frequencies of vibration. It follows that the following inertia terms equal zero: $\ddot{u}_o$, $\ddot{v}_o$, $\ddot{u}_{o,x}$, and $\ddot{v}_{o,y}$. Bowlus (2) and Palardy (15) determined that the effect of rotary inertia was negligible for the vibration of flat plates using Mindlin shear theory. Linneman (10) found the same was true using the higher order shear theory for first mode analysis. However, as its effect becomes more important for higher modes, it will be included in this general development. Finally, all time dependencies were assumed harmonic. This allows the time dependence to factor out of all the equations.

Implementing these assumptions in the preceding paragraphs results in the simplification of the equations of motion and associated boundary conditions to the following:

corresponding to u_o,

$$\int_0^b \int_0^a (N_{1,x} + N_{6,y} - \frac{1}{2R}M_{6,y})\,\delta u_o \, dxdy$$
$$+ \int_0^b N_1 \delta u_o \Big|_{x=0}^{x=a} dy + \int_0^a (N_6 - \frac{1}{2R}N_6)\,\delta u_o \Big|_{y=o}^{y=b} dx=0 \tag{51}$$

corresponding to v_o:

$$\int_0^b \int_0^a (\overline{I}_2'\omega^2\psi_y - \overline{I}_3'\omega^2 w_{,y} + N_{2,y} + N_{6,x} + \frac{1}{2R}M_{6,x})\,\delta v_o \, dxdy$$
$$+ \int_0^b (N_6 + \frac{1}{2R}M_6)\,\delta v_o \Big|_{x=0}^{x=a} dy + \int_0^a N_2 \delta v_o \Big|_{y=0}^{y=b} dx = 0 \tag{52}$$

corresponding to w:

$$\int_0^b \int_0^a [\overline{I}_5\omega^2(\psi_{x,x} + \psi_{y,y}) - k^2 I_7 \omega^2 (w_{,xx} + w_{,yy}) + I_1\omega^2 w + Q_{1,x}$$
$$- k(P_{1,xx} + P_{2,yy} + 2P_{6,xy}) + Q_{2,y} + 3k(R_{2,y} + R_{1,x})$$
$$- \frac{1}{R}[N_2 - k(L_{2,yy} + L_{6,xy})]\ \overline{N}_1 w_{,xx} + 2\overline{N}_6 w_{,xy} - \overline{N}_2(\frac{1}{R} - w_{,yy})]\,\delta w dxdy$$
$$+ \int_0^b [-k(P_{1,x} + 2P_{6,y}) + Q_1 + 3kR_1 + \frac{1}{R}kL_{6,y} + \overline{N}_1 w_{,x} + \overline{N}_6 w_{,y}]\,\delta w \Big|_{x=0}^{x=a} dy$$
$$+ \int_0^a [-k(P_{2,y} + 2P_{6,x}) + Q_2 + 3kR_2 + \frac{1}{R}k(L_{2,y} + L_{6,x}) + \overline{N}_2 w_{,y}$$
$$+ \overline{N}_6 w_{,x}]\,\delta w \Big|_{y=0}^{y=b} + k(2P_6 - \frac{1}{R}L_6)\,\delta w \Big|_{y=0}^{y=b} \Big|_{x=0}^{x=a} = 0 \tag{53}$$

corresponding to Ψ_x:

$$\int_0^b \int_0^a [\overline{I}_4\omega^2\psi_x - \overline{I}_5\omega^2 w,_x + k(P_{1,x} + P_{6,y}) + M_{1,x} + M_{6,y} - Q_1 - 3kR_1$$

$$- \frac{1}{R}(S_{6,y} + kL_{6,y})]\,\delta\psi_x dxdy \;+ \int_0^b [M_1 + 2kP_1]\,\delta\psi_x \Big|_{x=0}^{x=a} dy$$

$$+ \int_0^a [M_6 + kP_6 - \frac{1}{R}(S_6 + kL_6)]\,\delta\psi_y \Big|_{y=0}^{y=b} dx = 0 \tag{54}$$

corresponding to Ψ_y:

$$\int_0^b \int_0^a [\overline{I}_4\omega^2\psi_y - \overline{I}_5\omega^2 w,_y + k(P_{1,y} + P_{6,x}) + M_{1,y} + M_{6,x} - Q_2 - 3kR_2$$

$$- \frac{1}{R}(S_{2,y} + kL_{2,y})]\,\delta\psi_y dxdy \;+ \int_0^b [M_6 + kP_6]\,\delta\psi_x \Big|_{x=0}^{x=a} dy$$

$$+ \int_0^a [M_2 + 2kP_2 - \frac{1}{R}(S_2 + kL_2)]\,\delta\psi_y \Big|_{y=0}^{y=b} dx = 0 \tag{55}$$

Finally, the resultant quantities of Eq (33) and strain-displacement equations Eqs (7) are substituted into Eqs (51) through Eq (55) with the use of MACSYMA (17, 26) to obtain the final forms of the equations of motion and boundary conditions.

Equation (51) corresponding to δu_o becomes:

$$\int_0^b \int_0^a \{2A_{16}u_{o,xy} + A_{11}u_{o,xx} + A_{66}u_{o,yy} + A_{16}v_{o,xx} + (A_{12} + A_{66})v_{o,xy}$$

$$+ A_{26}v_{o,yy} + kE_{11}w,_{xxx} + 3kE_{16}w,_{xxy} + (E_{12} + 2E_{66})kw,_{xyy}$$

$$+ kE_{26}w,_{yyy} + (kE_{11} + B_{11})\psi_{x,xx} + 2(kE_{16}\ B_{16})\psi_{x,xy}$$

$$+ (kE_{66} + B_{66})\psi_{x,yy} + (kE_{16} + B_{16})\psi_{y,xx} + [k(E_{12} + E_{66})$$

$$+ B_{12} + B_{66}]\psi_{y,xy} + (kE_{26} + B_{26})\psi_{y,yy} + \frac{1}{R}\{-B_{16}u_{o,xy}$$

$$+ (\frac{1}{4R}D_{66} - B_{66})u_{o,yy} + \frac{1}{2}B_{16}v_{o,xx} - \frac{1}{4R}D_{66}v_{o,xy} + \frac{1}{2}B_{26}v_{o,yy}$$

$$+ A_{12}w,_{x} + [-k(F_{12} + 2F_{66}) + \frac{1}{2R}kG_{16}]w,_{xyy} - \frac{3}{2}kF_{16}w,_{xxy}$$

$$+ (A_{26} - \frac{1}{2R}B_{26})x,_{y} + \frac{k}{2}(\frac{1}{R}G_{26} - 3F_{26})w,_{yyy} - \frac{3}{2}(kF_{16}$$

$$+ D_{16})\psi_{x,xy} + [-\frac{3}{2}(kF_{22} - D_{22}) + \frac{1}{2R}(kG_{66} + E_{66})]\psi_{x,yy}$$

$$-[\frac{1}{2}(kF_{66} + D_{66}) + (kF_{12} + D_{12})]\psi_{y,xy} + [-\frac{3}{2}(kF_{26} + D_{26})$$

$$+ \frac{1}{2R}(kG_{26} + E_{26})]\psi_{y,yy}\Big\}\delta u_o\, dxdy$$

$$+ \int_0^b A_{11}u_{o,x} + A_{16}u_{o,y} + A_{16}v_{o,x} + A_{12}v_{o,y} + kE_{11}w,_{xx} + 2kE_{16}w,_{xy}$$

$$+ kE_{12}w,)yy + (kE_{11} + B_{11})\psi_{x,x} + (kE_{16} + B_{16})\psi_{x,y}$$

$$+ (kE_{16} + B_{16})\psi_{y,x} + (kE_{12} + B_{12})\psi_{y,y} + \frac{1}{R}\Big\{-\frac{1}{2}B_{16}u_{o,y}$$

$$+ \frac{1}{2}B_{16}v_{o,x} + A_{12}w - F_{16}w,_{xy} - kF_{12}w,_{yy} - (kF_{16} + D_{16})\psi_{x,y}$$

$$- (kF_{12} + D_{12})\psi_{y,y} \Big|_{x=0}^{x=a} \delta u_o\, dy$$

$$
\begin{aligned}
&+ \int_0^a A_{16}u_{o,x} + A_{66}u_{o,y} + A_{66}v_{o,x} + A_{26}v_{o,y} + kE_{16}w,_{xx} + 2kE_{66}w,_{xy} \\
&\quad + kE_{26}w,_{yy} + (kE_{16} + B_{16})\psi_{x,x} + (kE_{66} + B_{66})\psi_{x,y} + (kE_{66} \\
&\quad + B_{66})\psi_{y,x} + (kE_{26} + B_{26})\psi_{y,y} + \frac{1}{R}\Big\{-\frac{1}{2}B_{16}u_{o,x} + (B_{66} \\
&\quad + \frac{1}{4R}D_{66})u_{o,y} - \frac{1}{4R}D_{66}v_{o,x} - \frac{1}{2}B_{26}v_{o,y} - \frac{1}{2R}B_{26}w \\
&\quad - \frac{1}{2R}kF_{16}w,_{xx} + 2k(\frac{1}{4R}G_{66} - F_{66})w,_{xy} + \frac{k}{2}(\frac{1}{R}G_{26} \\
&\quad + 3kF_{26})w,_{yy} - \frac{1}{2}(kF_{16} + D_{16})\psi_{x,x} + [-\frac{3}{2}(kF_{66} + D_{66}) \\
&\quad + \frac{1}{2R}(kG_{66} + E_{66})]\psi_{x,y} - \frac{1}{2}(kF_{66} - D_{66})\psi_{y,x}\ [-\frac{3}{2}(kF_{26} \\
&\quad + D_{26}) + \frac{1}{2R}(kG_{26} + E_{26})\psi_{y,y}\Big|_{y=0}^{y=b} dx = 0
\end{aligned}
\tag{56}
$$

corresponding to δv_o:

$$\int_0^b \int_0^a A_{16}u_{o,xx} + (A_{12} + A_{66})\, u_{o,xy} + A_{66}v_{o,xx} + 2A_{26}v_{o,xy} + A_{22}v_{o,yy}$$

$$- \overline{I}_3'\omega^2 w,_y + kE_{16}w,_{xxx} + k(E_{12} + 2E_{66})\, w,_{xxy} + 3kE_{26}w,_{xyy} + kE_{22}w,_{yyy}$$

$$+ (kE_{16} + B_{16})\,\psi_{x,xx} + [(kE_{12} + B_{12}) + (kE_{66} + B_{66})]\,\psi_{x,xy} + (kE_{26}$$

$$+ B_{26})\,\psi_{x,yy} + \overline{I}_2'\omega^2\psi_y + (kE_{66} + B_{66})\,\psi_{y,xx} + 2(kE_{26} + B_{26})\,\psi_{y,xy}$$

$$+ (kE_{22} + B_{22})\,\psi_{y,yy} + \frac{1}{R}\left\{\frac{1}{2}B_{16}u_{o,xx} - \frac{1}{4R}D_{66}u_{o,xy} - \frac{1}{2}B_{26}u_{o,yy}\right.$$

$$+ (B_{66} + \frac{1}{4R}D_{66})\, v_{o,xx} + B_{26}v_{o,xy} + (A_{26} + \frac{1}{2R}B_{26})\, w,_x + A_{22}w,_y$$

$$+ \frac{k}{R}F_{16}w,_{xxx} - \frac{k}{2R}G_{66}w,_{xxy} - \frac{1}{2}(3F_{26} + \frac{1}{R}G_{26})\, w,_{xyy} - kF_{22}w,_{yyy}$$

$$+ \frac{1}{2}(kF_{16} + D_{16})\,\psi_{x,xx} - \frac{1}{2}[(kF_{66} + D_{66}) + \frac{1}{R}(kG_{66} + E_{66})]\,\psi_{x,xy}$$

$$- (kF_{26} + D_{26})\,\psi_{x,yy} + + \frac{1}{2}(kF_{66} + D66)\, osu_{y,xx} - \frac{1}{2}[(kF_{26} + D_{26})$$

$$\left. + \frac{1}{2R}(kG_{26} + E_{26})]\,\psi_{y,xy} - (kF_{22} + D_{22})\,\psi_{y,yy}\right\}\delta u_o\, dxdy$$

$$+ \int_0^b A_{16}u_{o,x} + A_{66}u_{o,y} + A_{66}v_{o,x} + A_{26}v_{o,y} + kE_{16}w,_{xx} + 2kE_{66}w,_{xy}$$

$$+ kE_{26}w,_{yy} + (kE_{16} + B_{16})\,\psi_{x,x} + (kE_{66} + B_{66})\,\psi_{x,y} + (kE_{66} + B_{66})\,\psi_{y,x}$$

$$+ (kE_{26} + B_{26})\,\psi_{y,y} + \frac{1}{R}\left\{\frac{1}{2}B_{16}u_{o,x} - \frac{1}{4R}D_{66}u_{o,y} + (B_{66} + \frac{1}{4R}D_{66})\, v_{o,x}\right.$$

$$+ \frac{1}{2}B_{26}v_{o,y} + (a_{26} + \frac{1}{2R}B_{26})\, w + \frac{1}{2}kF_{16}w,_{xx} - \frac{1}{2R}G_{66}w,_{xy} - \frac{k}{2}(F_{26}$$

$$+ \frac{1}{R}G_{26})\, w,_{yy} + \frac{1}{2}(kF_{16} + D_{16})\,\psi,x,x - \frac{1}{2}[(kF_{66} + D_{66}) + \frac{1}{R}(kG_{66}$$

$$+ E_{66})]\,\psi_{x,y} + \frac{1}{2}(kF_{66} + D_{66})\,\psi_{y,x} - \frac{1}{2}[(kF_{26} + D_{26}) + \frac{1}{R}(kG_{26}$$

$$\left. + E_{26})]\,\psi_{y,y}\right\}\Big|_{x=0}^{x=a}\delta v_o\, dy$$

$$+\int_0^a A_{12}u_{o,x} + A_{26}u_{o,y} + A_{26}v_{o,x} + A_{22}v_{o,y} + kE_{12}w_{,xx} + 2kE_{26}w_{,xy} + kE_{22}w_{,y}$$

$$+ (kE_{12} + B_{12})\psi_{x,x} + (kE_{26} + B_{26})\psi_{x,y} + (kE_{26} + B_{26})\psi_{y,x} + (kE_{22}$$

$$+ B_{22})\psi_{y,y} + \frac{1}{R}\left\{-\frac{1}{2}B_{26}u_{o,y} + \frac{1}{2}B_{26}v_{o,x} + A_{22}w - kF_{26}w_{,xy} - kF_{22}w_{,yy}\right.$$

$$\left. - (kF_{26} + D_{26})\psi_{x,y} - (kF_{22} + D_{22})\psi_{y,y}\right\} \Big|_{y=0}^{y=b} dx \tag{57}$$

corresponding to δw:

$$\int_0^b \int_0^a \{\overline{N}_1 w,xx + \overline{N}_2 w,yy + \overline{N}_6 w_{,xy} - k^2 I_7 \omega^2 (w_{,xx} + w_{,yy}) + I_1\omega^2 w$$

$$+ \overline{I}_5\omega^2(\psi_{x,x} + \psi_{y,y}) - kE_{11}u_{o,xxx} - 3kE_{16}u_{o,xxy} + k(2E_{66} + E_{12})u_{o,xyy}$$

$$- kE_{26}u_{o,yyy} - kE_{16}v_{o,xxx} + k(2E_{66} + E_{12})v_{o,xxy} - 3kE_{26}v_{o,xxy}$$

$$- kE_{22}v_{o,yy} + (9k^2F_{55} + 6kD_{55} + A_{55})w_{,xx} + (18k^2F_{45} + 12kD_{45}$$

$$+ 2A_{45})w_{,xy} + (9k^2F_{44} + 6kD_{44} + A_{44})w_{,yy} - k^2H_{11}w_{,xxxx}$$

$$- 4k^2H_{16}w_{,xxxy} - 2k^2(2H_{66} + H_{12})w_{,xxyy} - 4k^2H_{26}w_{,xyyy} - k^2H_{22}w_{,yyyy}$$

$$+ (9k^2F_{55} + 6kD_{55} + A_{55})\psi_{x,x} + (9k^2F_{45} + 6kD_{45} + A_{45})\psi_{x,y}$$

$$+ (9k^2F_{45} + 6kD_{45} + A_{45})\psi_{y,x} + (9k^2F_{44} + 6kD_{44} + A_{44})\psi_{y,y} - k(kH_{11}$$

$$+ F_{11})\psi_{x,xxx} - 3k(kH_{16} + F_{16})\psi_{x,xxy} - k(2kH_{66} + kH_{12} + 2F_{66}$$

$$+ F_{12})\psi_{x,xyy} - k(kH_{26} + F_{26})\psi_{x,yyy} - k(kH_{16} + F_{16})\psi_{y,xxx} - k(2kH_{66}$$

$$+ kH_{12} + 2F_{66} + F_{12})\psi_{y,xxy} - 3k(kH_{26} + F_{26})\psi_{y,xyy} - k(kH_{22}$$

$$+ F_{22})\psi_{y,yyy} + \frac{1}{R}\left\{-N_2 - A_{12}u_{o,x} - A_{26}u_{o,y} + \frac{3}{2}F_{16}u_{o,xxy} + k(2F_{66}\right.$$

$$+ F_{12})u_{o,xyy} + \frac{3}{2}kF_{26}u_{o,yyy} - A_{26}v_{o,x} - A_{22}v_{o,y} - \frac{1}{2}kF_{16}v_{o,xxx}$$

$$+ \frac{3}{2}kF_{26}v_{o,xyy} + kF_{22}v_{o,yyy} - 2kE_{12}w_{,xx} - 4kE_{26}w_{,xy} - 2kE_{22}w_{,yy}$$

$$
\begin{aligned}
&+ 2k^2 I_{16} w,_{xxxy} + 2k^2 (2I_{66} + I_{12}) w,_{xxyy} + 6k^2 I_{26} w,_{xyyy} + 2k^2 I_{22} w,_{yyyy} \\
&- (kE_{12} + B_{12}) \psi_{x,x} - (kE_{26} + B_{26}) \psi_{x,y} - (kE_{26} + B_{26}) \psi_{y,x} - (kE_{22} \\
&+ B_{22}) \psi_{y,y} + 2k(kI_{16} + G_{16}) \psi_{x,xxy} + (3k^2 I_{66} + k^2 I_{12} + 3kG_{66} \\
&+ kG_{12}) \psi_{x,xyy} + 2k(kI_{26} + G_{26}) \psi_{x,yyy} + (k^2 I_{66} + k^2 I_{12} + kG_{66} \\
&+ kG_{12}) \psi_{y,xxy} + 4k(kI_{26} + G_{26}) \psi_{y,xyy} + 2k(kI_{22} + G_{22}) \psi_{y,yyy} \\
&+ \frac{1}{R} \left\{ \frac{1}{2} B_{26} (u_{o,y} - v_{o,x}) - \frac{1}{2} kG_{66} (u_{o,xyy} - v_{o,xxy}) - \frac{1}{2} kG_{26} (u_{o,yyy} \right. \\
&- v_{o,xyy}) - A_{22} w + 2kF_{26} w,_{xy} + 2kF_{22} w,_{yy} - k^2 J_{66} w,_{xxyy} - k^2 J_{26} w,_{xyyy} \\
&- k^2 J_{22} w,_{yyyy} + (kF_{26}\ D_{26}) \psi_{x,y} + (kF_{22} + D_{22}) \psi_{y,y} - k(kF_{66} \\
&+ H_{66}) \psi_{x,xyy} - k(kJ_{26}\ H_{26}) \psi_{x,yyy} - k(kJ_{26} + H_{26}) \psi_{y,xyy} - k(kJ_{22} \\
&\left. + H_{22}) \psi_{y,yyy} \right\} \Big\} \delta w \, dx dy \\
&+ \int_0^b \Big\{ \overline{N}_1 w,_x + \overline{N}_6 w,_y - kE_{11} u_{o,xx} + 3kE_{16} u_{o,xy} - 2kE_{66} u_{o,yy} - kE_{16} v_{o,xx} \\
&- k(2E_{66} + E_{12}) v_{o,xy} - 2kE_{26} v_{o,yy} + (9k^2 F_{55} + 6kD_{55} + A_{55}) w,_x \\
&+ (9K^2 F_{45} + 6kD_{45} + A_{45}) w,_y - k^2 H_{11} w,_{xxx} - 4k^2 H_{16} w,_{xxy} - k^2 (4H_{66} \\
&+ H_{12}) w,_{xyy} - 2k^2 H_{26} w,_{yyy} + (9k^2 F_{55} + 6kD_{55} + A_{55}) \psi_x + (9k^2 F_{45} \\
&+ 6kD_{45} + A_{45}) \psi_y - k(kH_{11} + F_{11}) \psi_{x,xx} - 3k(kH_{16} + F_{16}) \psi_{x,xy} \\
&- 2k(kH_{66} + F_{66}) \psi_{x,yy} - k(kH_{16} + F_{16}) \psi_{y,xx} - (2k^2 H_{66} + k^2 H_{12} \\
&+ 2kF_{66} + F_{12}) \psi_{y,xy} - 2k(kH_{26} + F_{26}) \psi_{y,yy} + \frac{1}{R} \left\{ \frac{3}{2} kF_{16} u_{o,xy} \right. \\
&+ 2kF_{66} u_{o,yy} + \frac{1}{2} kF_{16} v_{o,xx} + kF_{26} v_{o,yy} - kE_{12} w,_x - 2kE_{26} w,_y \\
&+ 2k^2 I_{16} w,_{xxy} + k^2 (4I_{66}\ I_{12}) w,_{xyy} + 3k^2 I_{26} w,_{yyy} + 2k(kI_{16} \\
&+ G_{16}) \psi_{x.xy} + 6k(kI_{66} + G_{66}) \psi_{x,yy} + (k^2 I_{66} + k^2 I_{12} + 2kG_{66} \\
&+ 2kG_{12}) \psi_{u,xy} + 3k(kI_{26} + G_{26}) \psi_{y,yy} + \frac{1}{R} \left\{ -\frac{1}{2} kG_{66} u,_{o,yy} \right. \\
&- \frac{1}{2} kG_{66} v_{o,xy} + kF_{26} w,_y - k^2 J_{66} w,_{xyy} - k^2 J_{26} w,_{yyy} - k(kJ_{66} \\
&\left. + H_{66}) \psi_{x,yy} - k(kJ_{26} + H_{26}) \psi_{y,yy} \right\} \Big\} \, \delta w \Big|_{x=0}^{x=a} dy
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^a \{\overline{N}_2 w,_y + \overline{N}_6 w,_x - 2kE_{16}u_{o,xx} + k(2E_{66} + E_{12})u_{o,xy} - kE_{26}u_{o,yy} \\
&- 2kE_{66}v_{o,xx} - 3kE_{26}v_{o,xy} - kE_{22}v_{o,yy} + (9k^2F_{45} + 6kD_{45} + A_{45})w,_x \\
&+ (9k^2F_{44} + 6kD_{44} + A_{44})w,_y - 2k^2H_{16}w,_{xxx} - k^2(4H_{66} + H_{12})w,_{xxy} \\
&- 4k^2H_{26}w,_{xyy} - k^2H_{22}w,_{yyy} + (9k^2F_{45} + 6kD_{45} + A_{45})\psi_x + (9k^2F_{44} \\
&+ 6kD_{44} + A_{44})\psi_y - 2k(kH_{16} + F_{16})\psi_{x,xx} - (2k^2H_{66} + k^2H_{12} + 2kF_{66} \\
&+ kF_{12})\psi_{x,xy} - k(kH_{26} + F_{26})\psi_{x,yy} - 2k(kH_{66} + F_{66})\psi_{y,xx} - 3k(kH_{26} \\
&+ F_{26})\psi_{y,xy} - k(kH_{22} + F_{22})\psi_{y,yy} + \frac{1}{R}\{kF_{16}u_{o,xx} + k(2F_{66} + F_{12})u_{o,xy} \\
&+ \frac{3}{2}kF_{26}u_{o,yy} + \frac{3}{2}kF_{26}v_{o,xy} + kF_{22}v_{o,yy} - 2kE_{26}w,_x - kE_{22}w,_y \\
&+ k^2I_{16}w,_{xxx} + k^2(4I_{66} + I_{12})w,_{xxy} + 6k^2I_{26}w,_{xyy} + 2k^2I_{22}w,_{yyy} \\
&+ k(kI_{16} + G_{16})\psi_{x,xx} + (3k^2I_{66} + k^2I_{12} + 3kG_{66} + kG_{12})\psi_{x,xy} \\
&+ 2k(kI_{26} + G_{26})\psi_{x,yy} + k(kI_{66} + G_{66})\psi_{y,xx} + 4k(kI_{26} + G_{26})\psi_{y,xy} \\
&+ 2k(kI_{22} + G_{22})\psi_{y,yy} + \frac{1}{R}\left\{-\frac{1}{2}kG_{66}u_{o,xy} - \frac{1}{2}kG_{26}u_{o,yy} + \frac{1}{2}kG_{66}v_{o,xx}\right. \\
&+ \frac{1}{2}kG_{26}v_{o,xy} + kF_{26}w,_x + kF_{22}w,_y - k^2J_{66}w,_{xxy} - 2k^2J_{26}w,_{xyy} \\
&- k^2J_{22}w,_{yyy} - k(kJ_{66} + H_{66})\psi_{x,xy} - k(kJ_{26} + H_{26})\psi_{x,yy} - k(kJ_{26} \\
&+ H_{26})\psi_{y,xy} - k(kJ_{22} + H_{22})\psi_{y,yy}\}\}\}\,\delta w\Big|_{y=0}^{y=b} dx \\
&+ \{2kE_{16}u_{o,x} + 2kE_{66}u_{o,y} + 2kE_{66}v_{o,x} + 2kE_{26}v_{o,y} + 2k^2H_{16}w,_{xx} \\
&+ 4k^2H_{66}w,_{xy} + 2k^2H_{26}w,_{yy} + 2k(kH_{16} + F_{16})\psi_{x,x} + 2k(kH_{66} \\
&+ F_{66})\psi_{x,y} + 2k(kH_{66} + F_{66})\psi_{y,x} + 2k(kH_{26} + F_{26})\psi_{y,y} \\
&+ \frac{1}{R}\{-kF_{16}u_{o,x} - 2kF_{66}u_{o,y} - kF_{26}v,_{o,y} + 2kE_{26}w - 4k^2I_{66}w,_{xy} \\
&- k^2I_{16}w,_{xx} - 3k^2I_{26}w,_{yy} - k(kI_{16} + G_{16})\psi_{x,x} - 3k(kI_{66} + G_{66})\psi_{x,y} \\
&- k(kI_{66} + G_{66})\psi_{y,x} - 3k(kI_{26} + G_{26})\psi_{y,y} + \frac{1}{R}\left\{\frac{1}{2}kG_{66}(u_{o,y} - v_{o,x})\right. \\
&- kF_{26}w + k^2I_{66}w,_{xy} + k^2J_{26}w,_{yy} + k(kJ_{66} + H_{66})\psi_{x,y} + k(kJ_{26}
\end{aligned}
$$

$$+ kF_{26}w + k^2I_{66}w,_{xy} + k^2J_{26}w,_{yy} + k(kJ_{66} + H+66)\psi_{x,y} + k(kJ_{26}$$

$$+ H_{26})\psi_{y,y}\}\}\delta w \Big|_{y=0}^{y=b} \Big|_{x=0}^{x=a} = 0 \tag{58}$$

corresponding to $\delta\Psi_x$

$$\int_0^b\int_0^a \{\bar{I}_4\omega^2\psi_x - \bar{I}_5\omega^2 w,_x + k(E_{11} + B_{11})u_{o,xx} + 2(kE_{16} + B_{16})u_{o,xy} + (kE_{66}$$

$$+ B_{66})u_{o,yy} + (kE_{16} + B_{16})v_{o,xx} + (kE_{12} + kE_{66} + B_{12} + B_{66})v_{o,xy}$$

$$+ (kE_{26} + B_{26})v_{o,yy} - (9k^2F_{55} + 6kD_{55} + A_{55})w,_x - (9k^2F_{45} + 6kD_{45}$$

$$+ A_{45})w,_y + k(kH_{11} + F_{11})w,_{xxx} + 3k(kH_{16} + F_{16})w,_{xxy} (2k^2H_{66}$$

$$+ k^2H_{12} + 2kF_{66} + kF_{12})w,_{xyy} + k(kH_{26} + F_{26})w,_{yyy} - (9k^2F_{55} + 6kD_{55}$$

$$+ A_{55})\psi_x + (k^2H_{11} + 2kF_{11} + D_{11})\psi_{x,xx} + 2(k^2H_{16}\ 2kF_{16} + D_{16})\psi_{x,xy}$$

$$+ (k^2H_{66} + 2kF_{66} + D_{66})\psi_{,xyy} - (9k^2F_{45} + 6kD_{45} + A_{45})\psi_y + (k^2H_{16}$$

$$+ 2kF_{16} + D_{16})\psi_{y,xx} + (k^2H_{66} + k^2H_{12} + 2kF_{66} + 2kF_{12} + D_{66}$$

$$+ D_{12})\psi_{y,xy} + (k^2H_{26} + 2kF_{26} + D_{26})\psi_{y,yy} + \frac{1}{R}\Big\{-\frac{3}{2}(kF_{16} + D_{16})u_{o,xy}$$

$$- \frac{3}{2}(kF_{66}\ D_{66})u_{o,yy} + \frac{1}{2}(kF_{16} + D_{16}) - \frac{1}{2}(kF_{66} + D_{66})v_{o,xy} - (kF_{26}$$

$$+ D_{26})v_{o,yy} + (kE_{12} + B_{12})w,_x + (kE_{26} + B_{26})w,_Y - 2k(kI_{16}$$

$$+ G_{16})w,_{xxy} - (3k^2I_{66} + k^2I_{12} + 3kG_{66} + kG_{12})w,_{xyy} - 2k(kI_{26}$$

$$+ G_{26})w,_{yyy} - 2(k^2I_{16} + 2kG_{16} + E_{16})\psi_{x,xy} - 2(k^2I_{66} + 2kG_{66}$$

$$+ e_{66})\psi_{x,yy} - (k^2I_{66} + k^2I_{12} + 2kG_{66} + 2kG_{12} + E_{66} + E_{12})\psi_{y,xy}$$

$$- 2(k^2I_{26} + 2kG_{26} + E_{26})\psi_{y,yy} + \frac{1}{R}\Big\{\frac{1}{2}(kG_{66} + E_{66})u_{o,yy} - \frac{1}{2}(kG_{66}$$

$$+ E_{66})v_{o,xy} - (kF_{26} + D_{26})w,_y + k(kH_{66} + H_{66})w,_{xyy} + k(kJ_{26}$$

$$+ H_{26})w,_{yyy} + (k^2J_{65} + kH_{66} + F_{66})\psi_{x,yy} + (k^2J_{26} + kH_{26}$$

$$+ F_{26})\psi_{y,yy}\}\}\delta\psi_x\, dxdy$$

$$
\begin{aligned}
&+\int_0^b \{(2kE_{11} + B_{11}) u_{o,x} + (2kE_{16} + B_{16}) u_{o,y} + (2kE_{16} + B_{16}) v_{o,x} + (2kE_1 \\
&+ B_{12}) v_{o,y} + k(2kH_{11} + F_{11}) w,_{xx} + 2k(2kH_{16} + F_{16}) w,_{xy} + k(2kH_{12} \\
&+ F_{12}) w,_{yy} + (2k^2H_{11} + 3kF_{11} + D_{11}) \psi_{x,x} + (2k^2H_{16} + 3kF_{16} + D_{16}) \psi_{x,} \\
&+ (2k^2H_{16} + 3kF_{16} + D_{16}) \psi_{y,x} + (2k^2H_{12} + 3kF_{12}\ D_{12}) \psi_{y,y} \\
&+ \frac{1}{R}\Big\{-(kF_{16} + \frac{1}{2}D_{16}) u_{u,y} + (kF_{16} + \frac{1}{2}D_{16}) v_{o,x} + (2kE_{12} + B_{12}) w \\
&- k(2kI_{16} + G_{16}) w,_{xy} - k(2kI_{12} + G_{12}) w,_{yy} - (2k^2I_{16} + 3kG_{16} \\
&+ E_{16}) \psi_{x,y} - (2k^2I_{12} + 3kG_{12} + E_{12}) \psi_{y,y}\Big\}\Big\} \delta\psi_x \Big|_{x=0}^{x=a} dy \\
&+\int_0^a \{(kE_{16} + B_{16}) u_{o,x} + (kE_{66} + B_{66}) u_{o,y} + (kE_{66} + B_{66}) v_{o,x} + (kE_{26} \\
&+ B_{26}) v_{o,y} + k(kH_{16} + F_{16}) w,_{xx} + 2k(kH_{66} + F_{66}) w,_{xy} + k(kH_{26} \\
&+ F_{26}) w,_{yy} + (k^2H_{16} + 2kF_{16} + D_{16}) \psi_{x,x} + (k^2H_{66} + 2kF_{66}\ D_{66}) \psi_{x,y} \\
&+ (k^2H_{66} + 2kF_{66} + D_{66}) \psi_{y,x} + (k^2H_{26} + 2kF_{26} + D_{26}) \psi_{y,y} \\
&+\frac{1}{R}\Big\{-(kF_{16}\ D_{16}) u_{o,x} - \frac{3}{2}(kF_{66} + D_{66}) u_{o,y} - \frac{1}{2}(kF_{66} + D_{66}) v_{o,x} \\
&- (kF_{26} + F_{26}) v_{o,y} + (kE_{26}\ B_{26}) w - k(kI_{16} + G_{16}) w,_{xx} - 3k(kI_{66} \\
&+ G_{66}) w,_{xy} - 2k(kI_{26} + G_{26}) w,_{yy} - (k^2I_{16} + 2kG_{16} + E_{16}) \psi_{x,x} \\
&- 2(k^2I_{66} + 2kG_{66} + E_{66}) \psi_{x,y} - (k^2I_{66} + 2kG_{66} + E_{66}) \psi_{y,x} - 2(k^2I_{26} \\
&+ 2kG_{26} + E_{26}) \psi_{y,y} + \frac{1}{R}\Big\{\frac{1}{2}(kG_{66} + E_{66}) u_{o,y} - \frac{1}{2}(kG_{66} + E_{66}) v_{o,x} \\
&- (kF_{26} + D_{26}) w + k(kJ_{66} + H_{66}) w,_{xy} + k(kJ_{26} + H_{26}) w,_{yy} + (k^2J_{66} \\
&+ 2kH_{66} + F_{66}) \psi_{x,y} + (k^2J_{26} + 2kH_{26} + F_{26}) \psi_{y,y}\Big\}\Big\}\Big\} \delta\psi_x \Big|_{y=o}^{y=b} dx = 0
\end{aligned}
\tag{59}
$$

corresponding to $\delta\Psi_y$

$$\int_0^b\int_0^a \{\overline{I}_4\omega^2\psi_y - \overline{I}_5\omega^2 w,_y + (kE_{16} + B_{16})u_{o,xx} + (kE_{66} + kE_{12} + B_{66}$$

$$+ B_{12})u_{o,xy} + (kE_{26} + B_{26})u_{o,yy} + (kE_{66} + B_{66})v_{o,xx} + 2(kE_{26}$$

$$+ B_{26})v_{o,xy} + (kE_{22} + B_{22})v_{o,yy} - (9k^2F_{45} + 6kD_{45} + A_{45})w,_x$$

$$- (9k^2F_{44} + 6kD_{44} + A_{44})w,_y + k(kH_{16} + F_{16})w,_{xxx} + (2k^2H_{66} + k^2H_{12}$$

$$+ 2kF_{66} + kF_{12})w,_{xxy} + 3k(kH_{26}\ F_{26})w,_{xyy} + k(kH_{22} + F_{22})w,_{yyy}$$

$$- (9k^2F_{45} + 6kD_{45} + A_{45})\psi_x - (9k^2F_{44} + 6kD_{44} + A_{44})\psi_y + (k^2H_{16}$$

$$+ 2kF_{16} + D_{16})\psi_{x,xx} + (k^2H_{66} + k^2H_{12} + 2kF_{66} + 2kF_{12} + D_{66}$$

$$+ D_{12})\psi_{x,xy} + (k^2H_{26} + 2kF_{26} + D_{26})\psi_{x,yy} + (k^2H_{66} + 2kF_{66}$$

$$+ D_{66})\psi_{y,xx} + 2(k^2H_{26} + 2kF_{26} + d_{26})\psi_{y,xy} + (k^2H_{22} + 2kF_{22}$$

$$+ D_{22})\psi_{y,yy} + \frac{1}{R}\Big\{-\frac{1}{2}(kF_{66} + 2kF_{12} + D_{66} + 2D_{12})u_{o,xy} - \frac{3}{2}(kF_{26}$$

$$+ D_{26})u_{o,yy} + \frac{1}{2}(kF_{66} + D_{66})v_{o,xx} - \frac{1}{2}(kF_{26} + D_{26})v_{o,xy} - (kF_{22}$$

$$+ D_{22})v_{o,yy} + (kE_{26} + B_{26})w,_x + (kE_{22} + B_{22})w,_y - (k^2I_{66} + k^2I_{22}$$

$$+ kG_{66} + G_{12})w,_{xxy} - 4k(kI_{26} + G_{26})w,_{xyy} - 2k(kI_{22} + G_{22})w,_{yyy}$$

$$- (k^2I_{66} + k^2I_{12} + 2kG_{66} + 2kG_{12} + E_{66} + E_{12})\psi_{x,xy} - 2(k^2I_{26}$$

$$+ 2kG_{26} + E_{26})\psi_{x,yy} - 2(k^2I_{26} + 2kG_{26} + E_{26})\psi_{y,xy} - 2(k^2I_{22} + 2kG_{22}$$

$$+ E_{22})\psi_{y,yy} + \frac{1}{R}\Big\{\frac{1}{2}(kG_{26} + E_{26})u_{o,yy} - \frac{1}{2}(kG_{26} + E_{26})v_{o,xy} - (kF_{22}$$

$$+ D_{22})w,_y + k(kJ_{26} + H_{26})w,_{xyy} + k(kJ_{22} + H_{22})w,_{yyy} + (k^2J_{26} + 2kH_2$$

$$+ F_{26})\psi_{x,yy} + (k^2J_{22} + 2kH_{22}\ F_{22})\psi_{y,yy}\}\}\}\delta\psi_y\, dx dy$$

$$
\begin{aligned}
&+\int_0^b \{ (kE_{16} + B_{16}) u_{o,x} + (kE_{66} + B_{66}) u_{o,y} + (kE_{66} + B_{66}) v_{o,x} + (kE_{26} \\
&+ B_{26}) v_{o,y} + k(kH_{16} + F_{16}) w,_{xx} + 2k(kH_{66} + F_{66}) w,_{xy} + k(kH_{26} \\
&+ F_{26}) w,_{yy} + (k^2H_{16} + 2kF_{16} + D_{16}) \psi_{x,x} + (k^2H_{66} + 2kF_{66} + D_{66}) \psi_{x,y} \\
&+ (k^2H_{66} + 2kF_{66} + D_{66}) \psi_{y,x} + (k^2H_{26} + 2kF_{26} + D_{66}) \psi_{y,y} \\
&+ \frac{1}{R}\Big\{ -\frac{1}{2} (kF_{66} + D_{66}) u_{o,y} + \frac{1}{2} (kF_{66} + D_{66}) v_{o,x} + (kE_{26} + B_{26}) w \\
&- k(kI_{66} + G_{66}) w,_{xy} - k(kI_{26} + G_{26}) w,_{yy} - (k^2I_{66} + 2kG_{66} + E_{66}) \psi_{x,y} \\
&- (k^2I_{26} + 2kG_{26} + E_{26}) \psi_{y,y}\Big\}\} \, \delta\psi_y \Big|_{x=0}^{x=a} dy \\
&+ \int_0^a \{ (2kE_{12} + B_{12}) u_{o,x} + (2kE_{26} + B_{26}) u_{o,y} + (2kE_{26} + B_{26}) v_{o,x} + (2kE_2 \\
&+ B_{22}) v_{o,y} + 2k(kH_{12} + F_{12}) w,_{xx} + 2k(kH_{26} + F_{26}) w,_{xy} + k(2kH_{22} \\
&+ F_{22}) w,_{yy} + (2k^2H_{12} + 3kF_{12} + D_{12}) \psi_{x,x} + (2k^2H_{26} + 3kF_{26} \\
&+ D_{26}) \psi_{x,y} + (2k^2H_{26} + 3kF_{26} + D_{26}) \psi_{y,x} + (2k^2H_{22} + 3kF_{22} \\
&+ D_{22}) \psi_{y,y} + \frac{1}{R}\Big\{ -(2kF_{12} + D_{12}) u_{o,x} - \frac{3}{2} (kF_{26} + D_{26}) u_{o,y} - (kF_{26} \\
&+ D_{26}) v_{o,x} - (2kF_{22} + D_{22}) v_{o,y} + (2kE_2 + B_{22}) w - k(2kI_{12} + G_{12}) w,_{xx} \\
&- 3k(2kI_{26} + G_{26}) w,_{xy} - 2k(2kI_{22} + G_{22}) w,_{yy} - (2k^2I_{12} + 3kG_{12} \\
&+ E_{12}) \psi_{x,x} + 2(2k^2I_{26} + 3kG_{26} + E_{26}) \psi_{x,y} - (2k^2I_{26} + 3kG_{26} \\
&+ E_{26}) \psi_{y,x} - 2(2k^2I_{22} + 3kG_{22} + E_{22}) \psi_{y,y} + \frac{1}{R}\{ (kG_{26} + E_{26}) u_{o,y} \\
&- (kG_{26} + E_{26}) v_{o,x} - (2kF_{22} + D_{22}) w + k(2k^2I_{26} + H_{26}) w,_{xy} \\
&+ k(2kJ_{22} + H_{22}) w,_{yy} + (2k^2J_{26} + 3kH_{26} + F_{26}) \psi_{x,y} + (2k^2J_{22} \\
&+ 3kH_{22} + F_{22}) \psi_{y,y}\}\Big\}\} \, \delta\psi_y \Big|_{y=0}^{y=b} dx = 0
\end{aligned}
\tag{60}
$$

If the radius of curvature approaches infinity, the equations of motion above reduce to those of a flat plate. For symmetric laminates, the extensional stiffness matrix $[B_{ij}]$ and the higher order stiffness matrices $[E_{ij}]$, $[G_{ij}]$, and $[I_{ij}]$ are zero. The resulting equations for u_o and v_o are then completely decoupled from bending. The equations will only contain extensional stiffness terms, A_{ij}, and spatial derivatives of u_o and v_o. Similarly, the equations for w, Ψ_x and Ψ_y will only contain bending and the remaining higher order stiffness terms. For nonsymmetric flat plates, the equations for u_o and v_o are independent of direct bending, but do contain the coupling and higher order terms.

The equations of motion and boundary conditions will now be used to solve for the natural frequencies and boundary conditions for the cylindrical shell using the Galerkin technique.

Galerkin Technique

The classical Galerkin Technique is an approximate technique commonly used to solve partial differential equations of motion of the form:

$$\int_x \int_y F(\zeta(x,y))\,\delta\zeta(x,y)\,dydx + \int_y BC1(\zeta(x,y))\,\delta\zeta(x,y)\big|_x dy + \int_x BC2(\zeta(x,y))\,\delta\zeta(x,y)\big|_y dx = 0 \tag{61}$$

where $\zeta(x,y)$ represents the degree of freedom, $F(\zeta(x,y))$ is the differential equation of motion, a function of $\zeta(x,y)$ and its spatial derivatives, and $BC1(\zeta(x,y))$ and $BC2(\zeta(x,y))$ are the associated boundary conditions, also functions of $\zeta(x,y)$ (11:235-237; 27:163-165).

The assumed solution takes the form

$$\zeta(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}\Phi_{mn}(x,y) \tag{62}$$

where A_{mn} are undetermined coefficients and $\phi(x,y)$ are known comparison functions. In this manner, the boundary values are automatically satisfied, and no longer enter the problem. The values of M and N chosen would depend on the degree of accuracy required for the problem.

Following the assumption of a solution for $\zeta(x,y)$, the variation of Eq (62) is taken with respect to the undetermined

coefficient A_{mn}, and the results substituted back into Eq (61). The result will be M × N equations of the form

$$\int_x \int_y (DEOM(A_{11}\Phi_{11}(x,y) + A_{12}\Phi_{12}(x,y) + \ldots + A_{mn}\Phi_{mn}(x,y))\delta A_{mn}dydx \tag{63}$$

Since the coefficients A_{mn} are arbitrary, the only way for the set of equations to be satisfied is that each integral identically equal zero. The M × N integral equations can then be solved simultaneously for the coefficients.

The Galerkin technique will now be applied to the equations developed in this thesis. Since there are five equations involved, five assumed solutions are required:.

$$\begin{aligned}
\Psi_x(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\bar{\Psi}_{x\,mn}(x,y) \\
\Psi_y(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn}\bar{\Psi}_{y\,mn}(x,y) \\
w(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} C_{mn}\bar{w}_{mn}(x,y) \\
u_o(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} E_{mn}\bar{u}_{mn}(x,y) \\
v_o(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} G_{mn}\bar{v}_{mn}(x,y)
\end{aligned} \tag{64}$$

where A_{mn}, B_{mn}, C_{mn}, E_{mn}, and G_{mn} are undetermined coefficients and the "bar" terms are assumed solutions.

Choosing comparison functions for a problem such as this would be extremely difficult, as the natural boundary conditions are very complicated. Therefore, admissible functions are chosen instead. Admissible functions satisfy only the geometric boundary conditions. This will require including the boundary conditions in the Galerkin formulation, where the use of comparison functions would allow them to be ignored.

Boundary Conditions

This section will outline the selection of the admissible functions. Three boundaries are considered, simply supported on all edges, clamped on all edges, and a combination of clamped and simple supports.

For the simply supported condition, the following boundary conditions are specified:

at $x = 0$ and $x = a$ $\quad\quad w = \Psi_y = 0$

at $y = 0$ and $y = b$ $\quad\quad w = \Psi_x = 0$

In addition, an S-2 type boundary as described in Jones (9:244) is used here to describe the normal and tangential displacements at the edges. Normal displacement is allowed, while the tangential displacement is zero. Specifically,

at $x = o$ and $x = a$ $\quad\quad u_o \neq 0$ and $v_o = 0$

at $y = 0$ and $y = b$ $\quad\quad u_o = 0$ and $v_o \neq 0.$

To meet these conditions, the following admissible functions were chosen:

$$
\begin{aligned}
\psi_x(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \\
\psi_y(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}) \\
w(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \\
u_o(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} E_{mn}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \\
v_o(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})
\end{aligned}
\tag{65}
$$

with the corresponding single terms associated with the variations:

$$
\begin{aligned}
\delta u_o &: \cos(\frac{p\pi x}{a})\sin(\frac{q\pi y}{b}) \\
\delta v_o &: \sin(\frac{p\pi x}{a})\cos(\frac{q\pi y}{b}) \\
\delta w &: \sin(\frac{p\pi x}{a})\sin(\frac{q\pi y}{b}) \\
\delta \psi_x &: \cos(\frac{p\pi x}{a})\sin(\frac{q\pi y}{b}) \\
\delta \psi_y &: \sin(\frac{p\pi x}{a})\cos(\frac{q\pi y}{b})
\end{aligned}
\tag{66}
$$

In the above relations, the values of m and n govern the number of terms in the Galerkin equations, while the values of p and q govern the number of equations.

The remaining work in actually deriving the Galerkin equations was performed through the computerized symbolic manipulation system MACSYMA (17; 26). Performing the remaining calculations by hand would have been difficult and time consuming. Appendix C contains a copy of the MACSYMA batch file used to generate the equations.

The admissible functions and the single term expressions for the variations of u, v, w, Ψ_x and Ψ_y are first substituted into the five equations of motion. At this point, $\bar{N}_2$ and $\bar{N}_6$ are assumed equal to zero. MACSYMA then performed the integration over the double and single integrals representing the equation of motion and corresponding edge conditions.

The results of each integration depend on the particular value of m, n, p, and q. From the choice of trigonometric admissible functions, the following integrals are known.

$$\int_0^a \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{p\pi x}{a}\right)dx = \begin{Bmatrix} 0 & m = p \\ \frac{a}{2} & m \neq p \end{Bmatrix}$$

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx = \begin{Bmatrix} 0 & m \neq p \\ \frac{a}{2} & m = p \end{Bmatrix}$$

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{p\pi x}{a}\right)dx = \begin{cases} 0 & m = p \\ 0 & m \neq p,\quad (m+p) \quad \text{eve.} \\ \dfrac{2am}{\pi(m^2 - p^2)} & m \neq p,\quad (m+p) \quad \text{odc} \end{cases}$$

Thus for the simply supported boundary, there are only two cases which yield nonzero results upon integration:

Case (1): $m = p$ and $n = q$

Case (2): $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd

The set of Galerkin equations generated for Case (1) are show below:

Equation (56) for u_o becomes:

$$\begin{aligned}
&- A_{mn}\{([(12\pi^2B_{66}a^3h^2 - 16\pi^2E_{66}a^3)q^2 + (12\pi^2B_{11}ab^2h^2 \\
&\quad - 16\pi^2E_{11}ab^2)p^2]R^2 + (24\pi^2F_{66}a^3 - 18\pi^2D_{66}a^3h^2)q^2R \\
&\quad + (6\pi^2E_{66}a^3h^2 - 8\pi^2G_{66}a^3)q^2)/(48a^2bh^2R^2)\} \\
&- B_{mn}\{([(12\pi^2B_{66} + 12\pi^2B_{12})a^2bh^2 + (-16\pi^2E_{66} - \\
&\quad - 16\pi^2E_{12})a^2b]pqR^2 + [(-6\pi^2D_{66} - 12\pi^2D_{12})a^2bh^2 \\
&\quad + (8\pi^2F_{66} + 16\pi^2F_{12}a^2b]pqR)\ /(48a^2bh^2R^2)\} \\
&- C_{mn}\ \{([(-32\pi^3E_{66} - 16\pi^3E_{12})a^2pq^2 - 16\pi^3E_{11}b^2p^3]R^2 \\
&\quad + [(32\pi^3F_{66} + 16\pi^3F_{12})a^2pq^2 - 12\pi A_{12}a^2b^2h^2p]R \\
&\quad - 8\pi^3G_{66}a^2pq^2)/(48a^2bh^2R^2)\}
\end{aligned}$$

$$- E_{mn}\{[(12\pi^2 A_{26}a^3h^2q^2 + 12\pi^2 A_{11}ab^2h^2p^2)R^2 - 12\pi^2 B_{66}a^3h^2q^2R$$
$$+ 3\pi^2 D_{66}a^3h^2q^2]/(48a^2bh^2R^2)\}$$
$$- G_{mn}\{[(12\pi^2 A_{66} + 12\pi^2 A_{12})a^2bh^2pqR^2$$
$$- 3\pi^2 D_{66}a^2bh^2pq]/(48a^2bh^2R^2)\} = 0 \qquad (67)$$

Equation (57) for v_o becomes:

$$- A_{mn}\{([(12\pi^2 B_{66} + 12\pi^2 B_{12})ab^2h^2 + (-16\pi^2 E_{66} - 16\pi^2 E_{12})ab^2]pqR^2$$
$$+ (8\pi^2 F_{66}ab^2 - 6\pi^2 D_{66}ab^2h^2)pqR + (8\pi^2 G_{66}ab^2$$
$$- 6\pi^2 E_{66}ab^2h^2)pq)/(48ab^2h^2R^2)\}$$
$$- B_{mn}\{([(12\pi^2 B_{22}a^2bh^2 - 16\pi^2 E_{22}a^2b)q^2 + (12\pi^2 B_{66}b^3h^2$$
$$- 16\pi^2 E_{66}b^3)p^2]R^2 + [(16\pi^2 F_{22}a^2b - 12\pi^2 D_{22}a^2bh^2)q^2$$
$$+ (6\pi^2 D_{66}b^3h^2 - 8\pi^2 F_{66}b^3)p^2]R)/(48ab^2h^2R^2)\}$$
$$- C_{mn}\{([(-32\pi^3 E_{66} - 16\pi^3 E_{12})b^2p^2q - 16\pi^3 E_{22}a^2q^3]R^2$$
$$+ (16\pi^3 F_{22}a^2q^3 - 12\pi A_{22}a^2b^2h^2q)R + 8\pi^3 G_{66}b^2p^2q)/(48ab^2h^2R^2)\}$$
$$- E_{mn}\{[12\pi^2 A_{66} + 12\pi^2 A_{12})ab^2h^2pqR^2 - 3\pi^2 D_{66}ab^2h^2pq]/(48ab^2h^2R^2)$$
$$- G_{mn}\{[(12\pi^2 A_{22}a^2bh^2q^2 + 12\pi^2 A_{66}b^3h^2p^2)R^2 + 12\pi^2 B_{66}b^3h^2p^2R$$
$$+ 3\pi^2 D_{66}b^3h^2p^2]/(48ab^2h^2R^2)\}$$
$$= -\{ab\overline{I}_2'/4\}\omega^2 B_{mn} + \{\pi aq\overline{I}_3'/4\}\omega^2 C_{mn} \qquad (68)$$

Equation (58) for w becomes:

$$
\begin{aligned}
&A_{mn}\Big\{\Big[\Big(\big[(24\pi^3F_{66}+12\pi^3F_{12})a^3b^2h^2+(-32\pi^3H_{66}\\
&\quad-16\pi^3H_{12})a^3b^2\big]p^2q^3+\big[(12\pi^3F_{11}ab^4h^2-16\pi^3H_{11}ab^4)p^4\\
&\quad+(-9\pi A_{55}a^3b^4h^4+72\pi D_{55}a^3b^4h^2-144\pi F_{55}a^3b^4)p^2\big]q\Big)R^2\\
&\quad+\Big(\big[(-36\pi^3G_{66}-12\pi^3G_{12})a^3b^2h^2+(48\pi^3I_{66}\\
&\quad+16\pi^3I_{12})a^3b^2\big]p^2q^3+(9\pi B_{12}a^3b^4h^4-12\pi E_{12}a^3b^4h^2)p^2q\Big)R\\
&\quad+(12\pi^3H_{66}a^3b^2h^2-16\pi^3J_{66}a^3b^2)p^2q\Big]/(36a^3b^3h^4pqR^2)\Big\}\\
&+B_{mn}\Big\{\Big(\Big[(12\pi^3F_{22}a^4bh^2-16\pi^3H_{22}a^4b)pq^4+\Big(\big[(24\pi^3F_{66}\\
&\quad+12\pi^3F_{12})a^2b^3h^2+(-32\pi^3H_{66}-16\pi^3H_{12})a^2b^3\big]p^3\\
&\quad+(-9\pi A_{44}a^4b^3h^4+72\pi D_{44}a^4b^3h^2-144\pi F_{44}a^4b^3)p\Big)q^2\Big]R^2\\
&\quad+\Big[(32\pi^3I_{22}a^4b-24\pi^3G_{22}a^4bh^2)pq^4+\Big(\big[(-12\pi^3G_{66}\\
&\quad-12\pi^3G_{12})a^2b^3h^2+(16\pi^3I_{66}+16\pi^3I_{12})a^2b^3\big]p^3\\
&\quad+(9\pi B_{22}a^4b^3h^4-12\pi E_{22}a^4b^3h^2)p\Big)q^2\Big]R+(12\pi^3H_{22}a^4bh^2\\
&\quad-16\pi^3J_{22}a^4b)pa^4+(12\pi F_{22}a^4b^3h^2\\
&\quad-9\pi D_{22}a^4b^3h^4)pq^2\Big)/(36a^3b^3h^4pqR^2)\Big\}\\
&+C_{mn}\Big\{\Big[\Big(-16\pi^4H_{22}a^4pq^5+\big[(-64\pi^4H_{66}-32\pi^4H_{12})a^2b^2p^3\\
&\quad+(-9\pi^2A_{44}a^4b^2h^4+72\pi^2D_{44}a^4b^2h^2-144\pi^2F_{44}a^4b^2)p\big]q^3\\
&\quad+\big[-16\pi^4H_{11}b^4p^5+(-9\pi^2A_{55}a^2b^4h^4+72\pi^2D_{55}a^2b^4h^2\\
&\quad-144\pi^2F_{55}a^2b^4)p^3\Big)R^2+\Big(32\pi^4I_{22}a^4pq^5+\big[(64\pi^4I_{66}\\
&\quad+32\pi^4I_{12})a^2b^2p^3-24\pi^2E_{22}a^4b^2h^2p\big]q^3-24\pi^2E_{12}a^2b^4h^2p^3q\Big)R\\
&\quad-16\pi^4J_{22}a^4pq^5+(24\pi^2F_{22}a^4b^2h^2p-16\pi^4J_{66}a^2b^2p^3)q^3\\
&\quad-9\pi A_{22}a^4b^4h^4pq\Big]/(36a^3b^3h^4pqR^2)\Big\}\\
&+E_{mn}\Big\{\Big(\big[(24\pi^3E_{66}+12\pi^3E_{12})a^3b^2h^2p^2q^3+12\pi^3E_{11}ab^4h^2p^4q\big]R^2\\
&\quad+\big[(-24\pi^3F_{66}-12\pi^3F_{12})a^3b^2h^2p^2q^3+9\pi A_{12}a^3b^4h^4p^2q\big]\Big)R
\end{aligned}
$$

$$+ 6\pi^3 G_{66} a^3 b^2 h^2 p^2 q^3) / (36 a^3 b^3 h^4 pqR^2) \Big\}$$

$$+ G_{mn} \Big\{ \big([12\pi^3 E_{22} a^4 b h^2 p q^4 + (24\pi^3 E_{66} + 12\pi^3 E_{12}) a^2 b^3 h^2 p^3 q^2] R^2$$

$$+ (9\pi A_{22} a^4 b^3 h^4 p q^2 - 12\pi^3 F_{22} a^4 b h^2 p q^4) R$$

$$- 6\pi^3 G_{66} a^2 b^3 h^2 p^3 q^2 \big) / (36 a^3 b^3 h^4 pqR^2) \Big\} = \Big\{ \pi b p \overline{I}_5 / 4 \Big\} \omega^2 A_{mn}$$

$$+ \Big\{ \pi a q \overline{I}_5 / 4 \Big\} \omega^2 B_{mn} - \Big\{ [16\pi^2 (a^2 q^2 + b^2 p^2) I_7$$

$$+ 9 a^2 b^2 h^4 I_1] / (36 a b h^4) \Big\} \omega^2 C_{mn} + \{ \pi^2 b p^2 / (4a) \} \overline{N}_1 C_{mn} \qquad (69)$$

Equation (59) for Ψ_x becomes:

$$- A_{mn} \Big\{ \big([(18\pi^2 D_{66} a^3 h^4 - 48\pi^2 F_{66} a^3 h^2 + 32\pi^2 H_{66} a^3) q^2$$

$$+ (18\pi^2 D_{11} a b^2 h^4 - 48\pi^2 F_{11} a b^2 h^2 + 32\pi^2 H_{11} a b^2) p^2$$

$$+ 18 A_{55} a^3 b^2 h^4 - 144 D_{55} a^3 b^2 h^2 + 288 F_{55} a^3 b^2] R^2$$

$$+ (-36\pi^2 E_{66} a^3 h^4 + 96\pi^2 G_{66} a^3 h^2 - 64\pi^2 I_{66} a^3) q^2 R +$$

$$+ (18\pi^2 F_{66} a^3 h^4 - 48\pi^2 H_{66} a^3 h^2 + 32\pi^2 J_{66} a^3) q^2 \big) / (72 a^2 b h^4 R^2) \Big\}$$

$$- B_{mn} \Big\{ \big([(18\pi^2 D_{66} + 18\pi^2 D_{12}) a^2 b h^4 + (-48\pi^2 F_{66} - 48\pi^2 F_{12}) a^2 b h^2$$

$$+ (32\pi^2 H_{66} + 32\pi^2 H_{12}) a^2 b] pqR^2 + [(-18\pi^2 E_{66} - 18\pi^2 E_{12}) a^2 b h^4$$

$$+ (48\pi^2 G_{66} + 48\pi^2 G_{12}) a^2 b h^2 + (-32\pi^2 I_{66}$$

$$- 32\pi^2 I_{12}) a^2 b] pqR \big) / (72 a^2 b h^4 R^2) \Big\}$$

$$- C_{mn} \Big\{ \Big[\big([(-48\pi^3 F_{66} - 24\pi^3 F_{12}) a^2 h^2 + (64\pi^3 H_{66} + 32\pi^3 H_{12}) a^2] p q^2$$

$$+ (32\pi^3 H_{11} b^2 - 24\pi^3 F_{11} b^2 h^2) p^3 + (18\pi A_{55} a^2 b^2 h^4 - 144\pi D_{55} a^2 b^2 h^2$$

$$+ 288\pi F_{55} a^2 b^2) p \big) R^2 + \big([(72\pi^3 G_{66} + 24\pi^3 G_{12}) a^2 h^2 + (-96\pi^3 I_{66}$$

$$- 32\pi^3 I_{12}) a^2] p q^2 + (24\pi E_{12} a^2 b^2 h^2 - 18\pi B_{12} a^2 b^2 h^4) p \big) R$$

$$+ (32\pi^3 J_{66} a^2 - 24\pi^3 H_{66} a^2 h^2) p q^2 \Big] / (72 a^2 b h^4 R^2) \Big\}$$

$$- E_{mn} \Big\{ \big([(18\pi^2 B_{66} a^3 h^4 - 24\pi^2 E_{66} a^3 h^2) q^2 + 18\pi^2 B_{11} a b^2 h^4$$

$$- 24\pi^2 E_{11} a b^2 h^2) p^2] R^2 + (36\pi^2 F_{66} a^3 h^2 - 27\pi^2 D_{66} a^3 h^4) q^2 R$$

$$+ (9\pi^2 E_{66} a^3 h^4 - 12\pi^2 G_{66} a^3 h^2) q^2 \big) / (72 a^2 b h^4 R^2) \Big\}$$

$$- G_{mn}\,\Big\{\Big([(18\pi^2 B_{66} + 18\pi^2 B_{12})a^2bh^4 + (-24\pi^2 E_{66} - 24\pi^2 E_{12})a^2bh^2]pqR^2 + (12\pi^2 F_{66}a^2bh^2 - 9\pi^2 D_{66}a^2bh^4)pqR + (12\pi^2 G_{66}a^2bh^2 - 9\pi^2 E_{66}a^2bh^4)pq\Big)/(72a^2bh^4R^2)\Big\} = 0$$

(70)

Equation (60) for Ψ_y becomes:

$$- A_{mn}\,\Big\{\Big([(18\pi^2 D_{66} + 18\pi^2 D_{12})ab^2h^4 + (-48\pi^2 F_{66} - 48\pi^2 F_{12})ab^2h^2 + (32\pi^2 H_{66} + 32\pi^2 H_{12})ab^2]pqR^2 + [(-18\pi^2 E_{66} - 18\pi^2 E_{12})ab^2h^4 + (48\pi^2 G_{66} + 48\pi^2 G_{12})ab^2h^2 + (-32\pi^2 I_{66} - 32\pi^2 I_{12})ab^2]pqR\Big)/(72ab^2h^4r^2)\Big\}$$
$$- B_{mn}\,\Big\{\Big([(18\pi^2 D_{22}a^2bh^4 - 48\pi^2 F_{22}a^2bh^2 + 32\pi^2 H_{22}a^2b)q^2 + (18\pi^2 D_{66}b^3h^4 - 48\pi^2 F_{66}b^3h^2 + 32\pi^2 H_{66}b^3)p^2 + 18A_{44}a^2b^3h^4 - 144D_{44}a^2b^3h^2 + 288F_{44}a^2b^3]R^2 + (-36\pi^2 E_{22}a^2bh^4 + 96\pi^2 G_{22}a^2bh^2 - 64\pi^2 I_{22}a^2b)q^2R + (18\pi^2 F_{22}a^2bh^4 - 48\pi^2 H_{22}a^2bh^2 + 32\pi^2 J_{22}a^2b)q^2\Big)/(72ab^2h^4R^2)\Big\}$$
$$- C_{mn}\,\Big\{\Big(\Big[(32\pi^3 H_{22}a^2 - 24\pi^3 F_{22}a^2h^2)q^3 + \Big([(-48\pi^3 F_{66} - 24\pi^3 F_{12})b^2h^2 + (64\pi^3 H_{66} + 36\pi^3 H_{12})b^2]p^2 + 18\pi A_{44}a^2b^2h^4 - 144\pi D_{44}a^2b^2h^2 + 288\pi F_{44}a^2b^2\Big)q\Big]R^2 + \Big[(48\pi^3 G_{22}a^2h^2 - 64\pi^3 I_{22}a^2)q^3 + \Big([(24\pi^3 G_{66} + 24\pi^3 G_{12})b^2h^2 + (-32\pi^3 I_{66} - 32\pi^3 I_{12})b^2]p^2 - 18\pi B_{22}a^2b^2h^4 + 24\pi E_{22}a^2b^2h^2\Big)q\Big]R + (32\pi^3 J_{22}a^2 - 24\pi^3 H_{22}a^2h^2)q^3 + (18\pi D_{22}a^2b^2h^4 - 24\pi F_{22}a^2b^2h^2)q\Big)/(72ab^2h^4R^2)\Big\}$$
$$- E_{mn}\,\Big\{\Big([(18\pi^2 B_{66} + 18\pi^2 B_{12})ab^2h^4 - (24\pi^2 E_{66} + 24\pi^2 E_{12})ab^2h^2]pqR^2 + [(-9\pi^2 D_{66} - 18\pi^2 D_{12})ab^2h^4 + (12\pi^2 F_{66}$$

$$+ 24\pi^2 F_{12})\, ab^2h^2]\, pqR\}/(72ab^2h^4R^2)\}$$

$$- G_{mn}\, \{\{[(18\pi^2 B_{22}a^2bh^4 - 24\pi^2 E_{22}a^2bh^2)\, q^2 + (18\pi^2 B_{66}b^3h^4$$

$$- 24\pi^2 E_{66}b^3h^2)\, p^2]\, R^2 + [(24\pi^2 F_{22}a^2bh^2 - 18\pi^2 D_{22}a^2bh^4)\, q^2$$

$$+ (9\pi^2 D_{66}b^3h^4 - 12\pi^2 F_{66}b^3h^2)\, p^2]\, R\}/(72ab^2h^4R^2)\} = 0 \tag{71}$$

The Galerkin equations for Case (2) are as follows.

Equation (56) for u_o becomes:

$$A_{mn}\, \{[(24\pi B_{16}ab^2h^2 - 32\pi E_{16}ab^2)\, m^2npR^2 + 24\pi F_{16}ab^2$$

$$- 18\pi D_{16}ab^2h^2)\, m^2nqR]\, /3\pi ab^2h^2R^2(p^2 - m^2)\,(q^2 - n^2)\}$$

$$+ B_{mn}\, \{\{[(12\pi B_{26}a^2bh^2 - 16\pi E_{26}a^2b)\, mn^2 + (12\pi B_{16}b^3h^2$$

$$- 16\pi E_{16}b^3)\, m^3]\, qR^2 + (24\pi F_{26}a^2b - 18\pi D_{26}a^2bh^2)\, mn^2qR$$

$$+ (6\pi E_{26}a^2bh^2 - 8\pi G_{26}a^2b)\, mn^2q\}/3\pi ab^2h^2R^2(p^2 - m^2)\,(q^2 - n^2)\}$$

$$+ C_{mn}\, \{\{(-16\pi^2 E_{26}a^2mn^3 - 48\pi^2 E_{16}b^2m^3n)\, qR^2 + [24\pi^2 F_{26}a^2mn^3$$

$$+ (24\pi^2 F_{16}b^2m^3 - 12A_{26}a^2b^2h^2m)\, n]\, qR + (6B_{26}a^2b^2h^2mn$$

$$- 8\pi^2 G_{26}a^2mn^3)\, q\}/3\pi ab^2h^2R^2(p^2 - m^2)\,(q^2 - n^2)\}$$

$$+ E_{mn}\, \{(24\pi A_{16}m^2nqR^2 - 12\pi B_{16}m^2nqR)\, /3\pi R^2(p^2 - m^2)\,(q^2 - n^2)\}$$

$$+ G_{mn}\, \{[(12\pi A_{26}a^2bh^2mn^2 + 12\pi A_{16}b^3h^2m^3)\, qR^2 + (6\pi B_{16}b^3h^2m^3$$

$$- 6\pi B_{26}a^2bh^2mn^2)\, qR]\, /3\pi ab^2h^2R^2(p^2 - m^2)\,(q^2 - n^2)\} = 0 \tag{72}$$

Equation (57) for v_o becomes

$$
\begin{aligned}
& A_{mn} \Big\{\big([(12\pi B_{26}a^3h^2 - 16\pi E_{26}a^3)n^3 + (12\pi B_{16}ab^2h^2 \\
& \quad - 16\pi E_{16}ab^2)m^2n]pR^2 + [(16\pi F_{26}a^3 - 12\pi D_{26}a^3h^2)n^3 \\
& \quad + (6\pi D_{16}ab^2h^2 - 8\pi F_{16}ab^2)m^2n]pR\big)/3\pi a^2bh^2R^2(p^2 - m^2)(q^2 - n^2) \\
& + B_{mn} \Big\{[(24\pi B_{26}a^2bh^2 - 32\pi E_{26}a^2b)mn^2pR^2 + (8\pi F_{26}a^2b \\
& \quad - 6\pi D_{26}a^2bh^2)mn^2pR + (8\pi G_{26}a^2b \\
& \quad - 6\pi E_{26}a^2bh^2)mn^2p]/3\pi a^2bh^2R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + C_{mn} \Big\{\big((-48\pi^2E_{26}a^2mn^3 - 16\pi^2E_{16}b^2m^3n)pR^2 + [24\pi^2F_{26}a^2mn^3 \\
& \quad + (-8\pi^2F_{16}b^2m^3 - 12A_{26}a^2b^2h^2m)n]pR + (8\pi^2G_{26}a^2mn^3 \\
& \quad - 6B_{26}a^2b^2h^2mn)p\big)/3\pi a^2bh^2R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + E_{mn} \Big\{[(12\pi A_{26}a^3h^2n^3 + 12\pi A_{16}ab^2h^2m^2n)pR^2 + (6\pi B_{16}ab^2h^2m^2n \\
& \quad - 6\pi B_{26}a^3h^2n^3)pR]/3\pi a^2bh^2R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + G_{mn} \Big\{[(8A_{26}bR^2 + 4B_{26}bR)mn^2p]/h^2R^2(p^2 - m^2)(q^2 - n^2)\Big\} = 0
\end{aligned}
$$

(73)

Equation (58) for w becomes:

$$
\begin{aligned}
&- A_{mn}\Big\{\Big[\Big((48\pi^2F_{26}a^3h^2 - 64\pi^2H_{26}a^3)n^3 + [(144\pi^2F_{16}ab^2h^2 \\
&\quad - 192\pi^2H_{16}ab^2)m^2 - 36A_{45}a^3b^2h^4 + 288D_{45}a^3b^2h^2 \\
&\quad - 576F_{45}a^3b^2]n\Big)pqR^2 + \Big((128\pi^2I_{26}a^3 - 96\pi^2G_{26}a^3h^2)n^3 \\
&\quad + [(128\pi^2I_{16}ab^2 - 96\pi^2G_{16}ab^2h^2)m^2 + 36B_{26}a^3b^2h^4 \\
&\quad - 48E_{26}a^3b^2h^2]n\Big)pqR + [(48\pi^2H_{26}a^3h^2 - 64\pi^2J_{26}a^3)n^3 \\
&\quad + (48F_{26}a^3b^2h^2 - 36D_{26}a^3b^2h^4)n]pq\Big]/9\pi a^2b^2h^4R^2(p^2 \\
&\quad - m^2)(q^2 - n^2)\Big\} \\
&- B_{mn}\Big\{\Big([(144\pi^2F_{26}a^2bh^2 - 192\pi^2H_{26}a^2b)mn^2 + (48\pi^2F_{16}b^3h^2 \\
&\quad - 64\pi^2H_{16}b^3)m^3 + (-36A_{45}a^2b^3h^4 + 288D_{45}a^2b^3h^2 \\
&\quad - 576F_{45}a^2b^3)mpqR^2 + [(256\pi^2I_{26}a^2b - 192\pi^2G_{26}a^2bh^2]mn^2 \\
&\quad + (36B_{26}a^2b^3h^4 - 48E_{26}a^2b^3h^2)m]pqR + (48\pi^2H_{26}a^2bh^2 \\
&\quad - 64\pi^2J_{26}a^2b)mn^2pq\Big)/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
&- C_{mn}\Big\{\Big[\Big([(-72\pi A_{45}a^2b^2h^4 + 576\pi D_{45}a^2b^2h^2 - 1152\pi F_{45}a^2b^2)m \\
&\quad - 256\pi^3H_{16}b^2m^3]n - 256\pi^3H_{26}a^2mn^3\Big)pqR^2 + [384\pi^3I_{26}a^2mn^3 \\
&\quad + (128\pi^3I_{16}b^2m^3 - 192\pi E_{26}a^2b^2h^2m)n]pqR + (96\pi F_{26}a^2b^2h^2mn \\
&\quad - 128\pi^3J_{26}a^2mn^3)pq\Big]/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
&- E_{mn}\Big\{\Big((48\pi^2E_{26}a^3h^2n^3 + 144\pi^2E_{16}ab^2h^2m^2n)pqR^2 + [(36A_{26}a^3b^2h^4 \\
&\quad - 72\pi^2F_{16}ab^2h^2m^2)n - 72\pi^2F_{26}a^3h^2n^3]pqR + (24\pi^2G_{26}a^3h^2n^3 \\
&\quad - 18B_{26}a^3b^2h^4n)pq\Big)/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
&- G_{mn}\Big\{[(144\pi^2E_{26}a^2bh^2mn^2 + 48\pi^2E_{16}b^3h^2m^3)pqR^2 + (24\pi^2F_{16}b^3h^2m^3 \\
&\quad + 36A_{26}a^2b^3h^4m - 72\pi^2F_{26}a^2bh^2mn^2)pqR + (18B_{26}a^2b^3h^4m \\
&\quad - 24\pi^2G_{26}a^2bh^2mn^2)pq]/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} = 0
\end{aligned}
\tag{74}
$$

Equation (59) for Ψ_x becomes:

$$
\begin{aligned}
& A_{mn}\{[(72\pi^2 D_{16}ab^2h^4 - 192\pi^2 F_{16}ab^2h^2 + 128\pi^2 H_{16}ab^2)m^2nqR^2 \\
&\quad + (-72\pi^2 E_{16}ab^2h^4 + 192\pi^2 G_{16}ab^2h^2 \\
&\quad - 128\pi^2 I_{16}ab^2)m^2nqR]/9\pi^2ab^2h^4R^2(p^2 - m^2)(q^2 - n^2)\} \\
& + B_{mn}\{([(36\pi^2 D_{26}a^2bh^4 - 96\pi^2 F_{26}a^2bh^2 + 64\pi^2 H_{26}a^2b)mn^2 \\
&\quad + (36\pi^2 D_{16}b^3h^4 - 96\pi^2 F_{16}b^3h^2 + 64\pi^2 H_{16}b^3)m^3 + (36A_{45}a^2b^3h^4 \\
&\quad - 288D_{45}a^2b^3h^2 + 576F_{45}a^2b^3)m]qR^2 + (-72\pi^2 E_{26}a^2bh^4 \\
&\quad + 192\pi^2 G_{26}a^2bh^2 - 128\pi^2 I_{26}a^2b)mn^2qR + (36\pi^2 F_{26}a^2bh^4 \\
&\quad - 96\pi^2 H_{26}a^2bh^2 + 64\pi^2 J_{26}a^2b)mn^2q)/9\pi^2ab^2h^4R^2(p^2 \\
&\quad - m^2)(q^2 - n^2)\} \\
& + C_{mn}\{[((64\pi^3 H_{26}a^2 - 48\pi^3 F_{26}a^2h^2)mn^3 + [(192\pi^3 H_{16}b^2 \\
&\quad - 144\pi^3 F_{16}b^2h^2)m^3 + (36\pi A_{45}a^2b^2h^4 - 288\pi D_{45}a^2b^2h^2 \\
&\quad + 576\pi F_{45}a^2b^2)m]n)qR^2 + ((96\pi^3 G_{26}a^2h^2 - 128\pi^3 I_{26}a^2)mn^3 \\
&\quad + [(96\pi^3 G_{16}b^2h^2 - 128\pi^3 I_{16}b^2)m^3 + (48\pi E_{26}a^2b^2h^2 \\
&\quad - 36\pi B_{26}a^2b^2h^4)m]n)qR + [(64\pi^3 J_{26}a^2 - 48\pi^3 H_{26}a^2h^2)mn^3 \\
&\quad + (36\pi D_{26}a^2b^2h^4 - 48\pi F_{26}a^2b^2h^2)mn]q]/9\pi^2ab^2h^4R^2(p^2 \\
&\quad - m^2)(q^2 - n^2)\} \\
& + E_{mn}\{[(72\pi^2 B_{16}ab^2h^4 - 96\pi^2 E_{16}ab^2h^2)m^2nqR^2 + (72\pi^2 F_{16}ab^2h^2 \\
&\quad - 54\pi^2 D_{16}ab^2h^4)m^2nqR)/9\pi^2ab^2h^4R^2(p^2 - m^2)(q^2 - n^2)\} \\
& + G_{mn}\{([(36\pi^2 B_{26}a^2bh^4 - 48\pi^2 E_{26}a^2bh^2)mn^2 + (36\pi^2 B_{16}b^3h^4 \\
&\quad - 48\pi^2 E_{16}b^3h^2)m^3]qR^2 + [(48\pi^2 F_{26}a^2bh^2 - 36\pi^2 D_{26}a^2bh^2)mn^2 \\
&\quad + (18\pi^2 D_{16}b^3h^4 - 24\pi^2 F_{16}b^3h^2)m^3]qR)/9\pi^2ab^2h^4R^2(p^2 \\
&\quad - m^2)(q^2 - n^2)\} = 0
\end{aligned}
\qquad (75)
$$

Equation (60) for Ψ_y becomes

$$
\begin{aligned}
& A_{mn}\Big\{\Big[\Big((36\pi^2 D_{26}a^3h^4 - 96\pi^2 F_{26}a^3h^2 + 64\pi^2 H_{26}a^3)n^3 + [(36\pi^2 D_{16}ab^2h^4 \\
&\quad - 96\pi^2 F_{16}ab^2h^2 + 64\pi^2 H_{16}ab^2)m^2 + 36A_{45}a^3b^2h^4 - 288D_{45}a^3b^2h^2 \\
&\quad + 576F_{45}a^3b^2]n\Big)pR^2 + (-72\pi^2 E_{26}a^3h^4 + 192\pi^2 G_{26}a^3h^2 \\
&\quad - 128\pi\,\text{[illegible]}_{26}a^3)n^3pR + (36\pi^2 F_{26}a^3h^4 - 96\pi^2 H_{26}a^3h^2 \\
&\quad + 64\pi^2 H_{26}a^3)n^3p\Big]/9\pi^2a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + B_{mn}\Big\{[(72\pi^2 D_{26}a^2bh^4 - 192\pi^2 F_{26}a^2bh^2 + 128\pi^2 H_{26}a^2b)mn^2pR^2 \\
&\quad + (-72\pi^2 E_{26}a^2bh^4 + 192\pi^2 G_{26}a^2bh^2 \\
&\quad - 128\pi^2 I_{26}a^2b)mn^2pR]/9\pi^2a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + C_{mn}\Big\{\Big[\Big((192\pi^3 H_{26}a^2 - 144\pi^3 F_{26}a^2h^2)mn^3 + [(64\pi^3 H_{16}b^2 \\
&\quad - 48\pi^3 F_{16}b^2h^2)m^3 + (36\pi A_{45}a^2b^2h^4 - 288\pi D_{45}a^2b^2h^2 \\
&\quad + 576\pi F_{45}a^2b^2)m]n\Big)pR^2 + [(192\pi^3 G_{26}a^2h^2 - 256\pi^3 I_{26}a^2)mn^3 \\
&\quad + (48\pi E_{26}a^2b^2h^2 - 36\pi B_{26}a^2b^2h^4)mn]pR + (64\pi^3 J_{26}a^2 \\
&\quad - 48\pi^3 H_{26}a^2h^2)mn^3p\Big]/9\pi^2a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + E_{mn}\Big\{\Big([(36\pi^2 B_{26}a^3h^4 - 48\pi^2 E_{26}a^3h^2)n^3 + (36\pi^2 B_{16}ab^2h^4 \\
&\quad - 48\pi^2 E_{16}ab^2h^2)m^2n]pR^2 + (72\pi^2 F_{26}a^{h^2} - 54\pi^2 D_{26}a^3h^4)n^3pR \\
&\quad + (18\pi^2 E_{26}a^3h^4 - 24\pi^2 G_{26}a^3h^2)n^3p\Big)/9\pi^2a^2bh^4R^2(p^2 \\
&\quad - m^2)(q^2 - n^2)\Big\} = 0 \\
& + G_{mn}\Big\{[(72\pi^2 B_{26}a^2bh^4 - 96\pi^2 E_{26}a^2bh^2)mn^2pR^2 + (24\pi^2 F_{26}a^2bh^2 \\
&\quad - 18\pi^2 D_{26}a^2bh^4)mn^2pR + (24\pi^2 G_{26}a^2bh^2 \\
&\quad - 18\pi^2 E_{26}a^2bh^4)mn^2p]/9\pi^2a^2bh^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} = 0
\end{aligned}
$$

(76)

The Galerkin equations for the clamped boundary condition were derived in the same method as for the simply supported. Admissible functions were required to meet the following boundary conditions for clamped edges on all four sides.

at x = 0 and x = a $\qquad w = \Psi_x = \Psi_y = 0$

at y = 0 and y = b $\qquad w = \Psi_x = \Psi_y = 0$

Similarly to the simple supported case, the boundaries are given a C-2 type clamped boundary (9:244):

at x = 0 and x = a $\qquad u_o \neq 0$ and $v_o = 0$

at y = 0 and y = b $\qquad u_o = 0$ and $v_o \neq 0$

The following admissible functions were chosen:

$$\begin{aligned}
\psi_x(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\
\psi_y(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \\
w(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} C_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\
u_o(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} E_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\
v_o(x,y) &= \sum_{m=1}^{M}\sum_{n=1}^{N} G_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)
\end{aligned} \tag{77}$$

with the corresponding single terms associated with the variations:

$$\delta u_o: \quad \cos\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)$$

$$\delta v_o: \quad \sin\left(\frac{p\pi x}{a}\right)\cos\left(\frac{q\pi y}{b}\right)$$

$$\delta w: \quad \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)$$

$$\delta \psi_x: \quad \cos\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)$$

$$\delta \psi_y: \quad \sin\left(\frac{p\pi x}{a}\right)\cos\left(\frac{q\pi y}{b}\right) \tag{78}$$

Again, the values of the indices m and n determine the number of terms in each equation, and p and q determine the number of equations.

The procedure for generating the Galerkin equations is identical to that outlined in the previous section. However, resulting from the different admissible functions chosen, four nonzero cases result from the integration:

Case (1): $m = p$ and $n = q$

Case (2): $m = p$ and $n \neq q$, $(n + q)$ odd

Case (3): $m \neq p$, $(m + p)$ odd and $n = q$

Case (4): $m \neq p$, $(m + p)$ odd and $n \neq q$, $(n + q)$ odd

Each of the above cases results in a set of Galerkin equations similar to those presented previously for the simply supported boundary condition. They have the same type of form

and contain similar terms. These are not included in this section for brevity, however for further detail, they are included in Appendix D.

The final boundary condition is that of a combination simple-clamped. Specifically, the curved edges of the panel, along $y = 0$ and b were clamped, and the straight edges, along $x = 0$ and a, were simply supported. This condition is similar to that described by Bowlus and Reams, though they looked at flat plates (2; 18). The boundary conditions to be satisfied are as follows;

at $x = 0$ and $x = a$ $\quad\quad w = \Psi_x = \Psi_y = 0$

at $y = 0$ and $y = b$ $\quad\quad w = 0$ and $\Psi_x = 0$

Again, S-2 and C-2 type boundaries are assumed, so that the displacement are described as follows:

at $x = 0$ and $x = a$ $\quad\quad u_o \neq 0$ and $v_o = 0$

at $y = 0$ and $y = b$ $\quad\quad u_o = 0$ and $v_o \neq 0$

The admissible functions chosen are

$$
\begin{aligned}
\psi_x(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\
\psi_y(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \\
w(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\
u_o(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} E_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\
v_o(x,y) &= \sum_{m=1}^{M} \sum_{n=1}^{N} G_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)
\end{aligned}
\tag{79}
$$

with the corresponding single terms associated with the variations

$$
\begin{aligned}
\delta u_o &: \cos\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \\
\delta v_o &: \sin\left(\frac{p\pi x}{a}\right) \cos\left(\frac{q\pi y}{b}\right) \\
\delta w &: \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \\
\delta \psi_x &: \cos\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \\
\delta \psi_y &: \sin\left(\frac{p\pi x}{a}\right) \cos\left(\frac{q\pi y}{b}\right)
\end{aligned}
\tag{80}
$$

Like the clamped condition, integration for the Galerkin equations results in four nonzero cases. Again, the resulting equations are included in Appendix D.

The Galerkin equations are now assembled into an eigenvalue problem, to be solved for the critical buckling loads and natural frequencies.

Computer Code

The computer program used to solve for the buckling loads and natural frequencies was based on programs by Linneman and Reams (10; 18). One program was written for each boundary condition, basically identical except for the Galerkin equations used. Appendix B contains a listing of the FORTRAN code and the corresponding Galerkin equations for each boundary condition. Each program is divided into three sections, the main program and two subroutines.

The main program reads as input the following data describing the laminate properties and configuration.

1) an integer flag: "1" indicates a vibration problem, "2" indicates a buckling problem.

2) a, the length of the panel in the x direction

3) b, the length in the y direction

4) R, the radius of curvature

5) h, the laminate thickness

6) NPLYS, the number of plies in the laminate

7) θ, orientation angle of each ply

8) E_1, Young's modulus in the 1 direction

9) E_2, Young's modulus in the 2 direction

10) G_{12}, the shear modulus in the 1-2 plane

11) ν_{12}, Poisson's ratio

12) ρ, the mass density

13) M = N, the maximum number of terms in each admissible function

All variables and arrays are declared double precision, and workspace is allocated for the eigenvalue calculations. The program as written can handle M and N values up to 10. Higher values would require much higher run times, and would be impractical for the computer facilities available for this work. For M = N = 10, there are 100 terms in each admissible function, resulting in 500 x 500 matrices for the eigenvalue problem.

The main program also calculates ν_{21}, G_{13}, and G_{23} using the following relationships

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1}$$

$$G_{13} = G_{12}$$

$$G_{23} = 0.8\ G_{12}$$

It then calls the first subroutine, called "LAMINAT", which uses the above data to calculate the extensional, coupling, and bending, and higher order stiffness matrices defined by Eq (34). It follows the procedure outlined in Section II, first calculating the reduced stiffness terms $[Q_{ij}]$ described by (26). It then applies the transformation relations, Eq (31), to find the transformed reduced stiffness terms, $[\bar{Q}_{ij}]$, for each ply. These terms are summed over the thickness of

the laminate, and using the definitions given by Eqs (34), the extensional, bending, coupling, and higher order stiffness terms are calculated. These values are then returned to the main program, where the second subroutine is called.

The subroutine, called GALERK, sets up the eigenvalue problem. It loops through the values of m, n, p, and q, and generates the appropriate Galerkin equations, based on the applicable integration case as outlined in the previous chapter. The results obtained for each loop are then compiled into matrix form, represented below:

$$\begin{bmatrix} Stiffness \\ terms \end{bmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ E_{mn} \\ G_{mn} \end{Bmatrix} = (\omega^2 \;,\; \overline{N}_1) \begin{bmatrix} Mass/Inertia \\ terms \end{bmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \\ E_{mn} \\ G_{mn} \end{Bmatrix}$$

Both the stiffness and mass/inertia matrices are (5*M*N) by (5*M*N). The terms ω^2 and $\overline{N}_1$ are the eigenvalues. The integer flag specified at the beginning of the program determines which is solved for. The mass/inertia matrix will contain those terms associated with whichever term is sought. The column vector is the eigenvector.

Appendix B contains the GALERK subroutines for each of the boundary conditions investigated.

The stiffness and mass/inertia matrices are then submitted to the IMSL subroutine DGVCRG, which solves for the eigenvalues and eigenvectors (8).

The remaining portion of the program also calculates and prints the deflections along the midlines of the laminate, thus giving the mode shape. Though not a focus of this thesis, this feature is a useful tool for future investigations.

III. *Discussion of Results*

Laminated Cylindrical Shell Properties. The material used for the cylindrical composite shell studied here is graphite/epoxy (AS/3501), which has the following material properties:

E_1 = 2.10E+07 psi

E_2 = 1.40E+06 psi

G_{12} = 6.00E+05 psi

ν_{12} = 0.3

ρ = 1.42454E-04 slugs/in^3

For each of the three boundary conditions, symmetric and antisymmetric ply lay-ups were investigated. These included sequences of [0/90], [+45/-45], and [0/+45/-45/90]. The symmetric lay-ups will be designated by a subscript s, and the antisymmetric by subscript as. For example, a symmetric lay-up of [0/45/-45/90/90/-45/45/0] would be referred to simply as $[0/45/-45/90]_s$. The corresponding stiffness terms are presented in Tables 1 through 5.

Table 1. Stiffness Elements for $[0/90]_s$ Laminate

Extensional Elements (lb/in)		
A_{11} = 11267605.634 A_{16} = 0.000 A_{44} = 540000.000	A_{12} = 422535.211 A_{26} = 0.000 A_{45} = 0.000	A_{22} = 11267605.634 A_{66} = 600000.000 A_{55} = 540000.000
Coupling Elements (lb·in/in)		
B_{11} = 0.000 B_{16} = 0.000	B_{12} = 0.000 B_{26} = 0.000	B_{22} = 0.000 B_{66} = 0.000
Bending Elements (lb·in)		
D_{11} = 1555164.319 D_{16} = 0.000 D_{44} = 42150.000	D_{12} = 35211.268 D_{26} = 0.000 D_{45} = 0.000	D_{22} = 322769.953 D_{66} = 50000.000 D_{55} = 48750.000
Higher Order Elements (lb·in^2), (lb·in^3), (lb·in^4), (lb·in^5), (lb·in^6), (lb·in^7)		
E_{11} = 0.000 E_{16} = 0.000	E_{12} = 0.000 E_{26} = 0.000	E_{22} = 0.000 E_{66} = 0.000
F_{11} = 256382.042 F_{16} = 0.000 F_{44} = 6046.875	F_{12} = 5281.690 F_{26} = 0.000 F_{45} = 0.000	F_{22} = 25308.099 F_{66} = 7500.000 F_{55} = 7453.125
G_{11} = 0.000 G_{16} = 0.000	G_{12} = 0.000 G_{26} = 0.000	G_{22} = 0.000 G_{66} = 0.000
H_{11} = 46814.088 H_{16} = 0.000	H_{12} = 943.159 H_{26} = 0.000	H_{22} = 3487.723 H_{66} = 1339.286
I_{11} = 0.000 I_{16} = 0.000	I_{12} = 0.000 I_{26} = 0.000	I_{22} = 0.000 I_{66} = 0.000
J_{11} = 9152.886 J_{16} = 0.000	J_{12} = 183.392 J_{26} = 0.000	J_{22} = 628.022 J_{66} = 260.417

Table 2. Stiffness Elements for the $[\pm 45]_s$ Laminate

Extensional Elements (lb/in)		
A_{11} = 6445070.414 A_{16} = 0.000 A_{44} = 540000.000	A_{12} = 5245070.423 A_{26} = 0.000 A_{45} = 0.000	A_{22} = 6445070.431 A_{66} = 5422535.211 A_{55} = 540000.000
Coupling Elements (lb·in/in)		
B_{11} = 0.000 B_{16} = 0.000	B_{12} = 0.000 B_{26} = 0.000	B_{22} = 0.000 B_{66} = 0.000
Bending Elements (lb·in)		
D_{11} = 537089.201 D_{16} = 308098.591 D_{44} = 45000.000	D_{12} = 437089.202 D_{26} = 308098.591 D_{45} = -3750.000	D_{22} = 537089.203 D_{66} = 451877.934 D_{55} = 45000.000
Higher Order Elements (lb·in^2), (lb·in^3), (lb·in^4), (lb·in^5), (lb·in^6), (lb·in^7)		
E_{11} = 0.000 E_{16} = 0.000	E_{12} = 0.000 E_{26} = 0.000	E_{22} = 0.000 E_{66} = 0.000
F_{11} = 80563.380 F_{16} = 57768.486 F_{44} = 6750.000	F_{12} = 65563.380 F_{26} = 57768.486 F_{45} = -703.125	F_{22} = 80563.380 F_{66} = 67781.690 F_{55} = 6750.000
G_{11} = 0.000 G_{16} = 0.000	G_{12} = 0.000 G_{26} = 0.000	G_{22} = 0.000 G_{66} = 0.000
H_{11} = 14386.318 H_{16} = 10831.591	H_{12} = 11707.746 H_{26} = 10831.591	H_{22} = 14386.318 H_{66} = 12103.873
I_{11} = 0.000 I_{16} = 0.000	I_{12} = 0.000 I_{26} = 0.000	I_{22} = 0.000 I_{66} = 0.000
J_{11} = 2797.340 J_{16} = 2131.216	J_{12} = 2276.506 J_{26} = 2131.216	J_{22} = 2797.340 J_{66} = 2353.531

Table 3. Stiffness Elements for the $[0/\pm45/90]_s$ Laminate

Extensional Elements (lb/in)		
A_{11} = 8856338.024 A_{16} = 0.000 A_{44} = 540000.000	A_{12} = 2833802.817 A_{26} = 0.000 A_{45} = 0.000	A_{22} = 885938.033 A_{66} = 5422535.211 A_{55} = 540000.000
Coupling Elements (lb·in/in)		
B_{11} = 0.000 B_{16} = 0.000	B_{12} = 0.000 B_{26} = 0.000	B_{22} = 0.000 B_{66} = 0.000
Bending Elements (lb·in)		
D_{11} = 1237852.112 D_{16} = 77024.648 D_{44} = 42187.500	D_{12} = 198474.178 D_{26} = 77024.648 D_{45} = -937.500	D_{22} = 313556.338 D_{66} = 213262.911 D_{55} = 47812.500
Higher Order Elements ($lb \cdot in^2$), ($lb \cdot in^3$), ($lb \cdot in^4$), ($lb \cdot in^5$), ($lb \cdot in^6$), ($lb \cdot in^7$)		
E_{11} = 0.000 E_{16} = 0.000	E_{12} = 0.000 E_{26} = 0.000	E_{22} = 0.000 E_{66} = 0.000
F_{11} = 220472.601 F_{16} = 10831.591 F_{44} = 6178.711	F_{12} = 19527.949 F_{26} = 10831.591 F_{45} = -131.836	F_{22} = 32725.022 F_{66} = 21746.259 F_{55} = 7321.289
G_{11} = 0.000 G_{16} = 0.000	G_{12} = 0.000 G_{26} = 0.000	G_{22} = 0.000 G_{66} = 0.000
H_{11} = 42782.777 H_{16} = 1297.534	H_{12} = 2379.401 H_{26} = 1297.534	H_{22} = 4646.550 H_{66} = 2775.528
I_{11} = 0.000 I_{16} = 0.000	I_{12} = 0.000 I_{26} = 0.000	I_{22} = 0.000 I_{66} = 0.000
J_{11} = 8691.133 J_{16} = 152.300	J_{12} = 340.545 J_{26} = 152.300	J_{22} = 775.469 J_{66} = 417.570

Table 4. Stiffness Elements for $[0/90]_{as}$ Laminate

Extensional Elements (lb/in)		
A_{11} = 11267605.634 A_{16} = 0.000 A_{44} = 540000.000	A_{12} = 422535.211 A_{26} = 0.000 A_{45} = 0.000	A_{22} = 11267605.634 A_{66} = 600000.000 A_{55} = 540000.000
Coupling Elements (lb·in/in)		
B_{11} = -1232394.366 B_{16} = 0.000	B_{12} = 0.000 B_{26} = 0.000	B_{22} = 1232394.366 B_{66} = 0.000
Bending Elements (lb·in)		
D_{11} = 938967.136 D_{16} = 0.000 D_{44} = 45000.000	D_{12} = 35211.268 D_{26} = 0.000 D_{45} = 0.000	D_{22} = 938967.136 D_{66} = 50000.000 D_{55} = 45000.000
Higher Order Elements (lb·in^2), (lb·in^3), (lb·in^4), (lb·in^5), (lb·in^6), (lb·in^7)		
E_{11} = -269586.268 E_{16} = 0.000	E_{12} = 0.000 E_{26} = 0.000	E_{22} = 269586.268 E_{66} = 0.000
F_{11} = 140845.070 F_{16} = 0.000 F_{44} = 6750.000	F_{12} = 5281.690 F_{26} = 0.000 F_{45} = 0.000	F_{22} = 140845.070 F_{66} = 7500.000 F_{55} = 6750.000
G_{11} = -49745.085 G_{16} = 0.000	G_{12} = 0.000 G_{26} = 0.000	G_{22} = 49745.085 G_{66} = 0.000
H_{11} = 25150.905 H_{16} = 0.000	H_{12} = 943.159 H_{26} = 0.000	H_{22} = 25150.905 H_{66} = 1339.286
I_{11} = -9552.662 I_{16} = 0.000	I_{12} = 0.000 I_{26} = 0.000	I_{22} = 9552.862 I_{66} = 0.000
J_{11} = 4890.454 J_{16} = 0.000	J_{12} = 183.392 J_{26} = 0.000	J_{22} = 4890.454 J_{66} = 260.417

Table 5. Stiffness Elements for the $[\pm 45]_{as}$ Laminate

Extensional Elements (lb/in)		
A_{11} = 6445070.414	A_{12} = 5245070.423	A_{22} = 6445070.431
A_{16} = 0.000	A_{26} = 0.000	A_{66} = 5422535.211
A_{44} = 540000.000	A_{45} = 0.000	A_{55} = 540000.000
Coupling Elements (lb·in/in)		
B_{11} = 0.000	B_{12} = 0.000	B_{22} = 0.000
B_{16} = -616197.183	B_{26} = -616197.184	B_{66} = 0.000
Bending Elements (lb·in)		
D_{11} = 537089.201	D_{12} = 437089.202	D_{22} = 537089.203
D_{16} = 0.000	D_{26} = 0.000	D_{66} = 451877.934
D_{44} = 45000.000	D_{45} = 0.000	D_{55} = 45000.000
Higher Order Elements $(lb\cdot in^2)$, $(lb\cdot in^3)$, $(lb\cdot in^4)$, $(lb\cdot in^5)$, $(lb\cdot in^6)$, $(lb\cdot in^7)$		
E_{11} = 0.000	E_{12} = 0.000	E_{22} = 0.000
E_{16} = -134793.134	E_{26} = -134793.134	E_{66} = 0.000
F_{11} = 80563.380	F_{12} = 65563.380	F_{22} = 80563.380
F_{16} = 0.000	F_{26} = 0.000	F_{66} = 67781.690
F_{44} = 6750.000	F_{45} = 0.000	F_{55} = 6750.000
G_{11} = 0.000	G_{12} = 0.000	G_{22} = 0.000
G_{16} = -24872.543	G_{26} = -24872.543	G_{66} = 0.000
H_{11} = 14386.318	H_{12} = 11707.746	H_{22} = 14386.318
H_{16} = 0.000	H_{26} = 0.000	H_{66} = 12103.873
I_{11} = 0.000	I_{12} = 0.000	I_{22} = 0.000
I_{16} = -4776.431	I_{26} = -4776.431	I_{66} = 0.000
J_{11} = 2797.340	J_{12} = 2276.506	J_{22} = 2797.340
J_{16} = 0.000	J_{26} = 0.000	J_{66} = 2353.531

The stiffness elements associated with the $[0/90]_s$ laminate are characterized by large extensional stiffnesses in the 1 and 2 directions, as all the fibers are oriented along one or the other direction. The extensional terms A_{16} and A_{26}

are zero, as this is a balanced laminate. This is true for all the laminates investigated. Additionally, the bending terms D_{16}, D_{26}, and D_{45} are all zero. D_{16} and D_{26} are terms representing twist coupling. These terms are generally present for other ply orientations.

The stiffness terms associated with the $[0/90]_{as}$ differ from those associated with the $[0/90]_s$ ply layup with the addition of terms such as B_{11} and B_{22}, E_{11} and E_{22}, G_{11} and G_{22}, and I_{11} and I_{22}. The B_{11} and B_{22} stiffnesses indicate bending-extension coupling is present. (The other terms are higher order and do not have a true physical representation.) Due to this coupling, the vibration frequencies and buckling loads are expected to be lower for the nonsymmetric case than for the symmetric case (9; 28).

The stiffness terms for the $[\pm 45]_s$ differ from the [0/90] layups primarily in several ways. First, they contain the twist coupling terms D_{16} and D_{26}. As Jones describes, the presence of these terms makes a closed form solution impossible, as the governing equations are not separable (9:263). Thus the presence of these terms may cause slower convergence. Additionally, the stiffness terms in the 1 and 2 directions of the fibers are generally smaller than for the $[0/90]_s$ laminate, as the 45 degree fibers provide less stiffness in these directions.

Finally the [0/±45/90] laminates also have a complete D_{ij} stiffness matrix, due to the presence of the 45 degree plies. The magnitudes of the extensional stiffness terms are in between the values for the $[0/90]_s$ and the $[\pm 45]_s$ laminates. Thus one might expect its behavior in buckling and vibration to also fall between these two cases.

Simply Supported Boundary Condition.

In first analyzing the results of this investigation, it is important to look at the convergence of the eigenvalues as more terms are included in the assumed solutions. As the Galerkin technique involves approximate solutions for the displacements, generally the natural boundary conditions for the panel will not be satisfied. Therefore, an exact solution cannot be found using this technique. A well chosen displacement field will tend to converge quickly, while a poor choice will converge slowly, if at all. Of course, absolute convergence cannot be proven, since an infinite number of terms would be required. Therefore, the rate of convergence is a good indication of the suitability of the assumed solutions.

This concept is ideally represented by the first boundary condition: simply supported on all edges. The solutions for the buckling loads and natural frequencies for the $[0/90]_s$ laminate are found to converge immediately at M = N = 2. The assumed solutions for this case correspond to an exact solution.

Figures 4 and 5 show the effect of curvature h/R on the buckling loads and frequencies. As pointed out in Chapter II, as the radius approaches zero, for a symmetric laminate, the membrane equations of motion for u_o and v_o are decoupled from bending. As the curvature increases, the membrane and bending

stiffnesses are coupled through the shell equations, Eqs (56) through (60). The panel becomes deeper and stiffer, and the buckling loads and frequencies increase.

The panels investigated in this section are all square, a = b = 1. The curvature is varied from h/R = 0 for a flat plate to h/R = 1/20. The ratio of the circumferential length to the thickness (b/R = a/R) is varied from 10 to 40.

Several important trends are identified. As expected, the buckling loads and frequencies are seen to increase as h/R is increased due to the membrane and bending coupling. Additionally, the effect of increasing the span to thickness ratio, b/h, is seen to lower the loads and frequencies. However, a major difference is evident. For the buckling problem, the effect of increasing the curvature for large spans, (effectively a thin shell) is quite pronounced. For a b/h value of 40, the buckling load for a thickness to radius of 1/20 is much greater than that for a flat plate of the same dimensions. The effect on the frequencies is comparatively slight. It would seem that for a deep shell, the effects of increasing the span have a large effect on the critical buckling load. It does not seem to have a significant effect on the vibration problem. Note that there is some behavior evident in Figure 4 of some values not following these trends. These values are circled, and assumed incorrect.

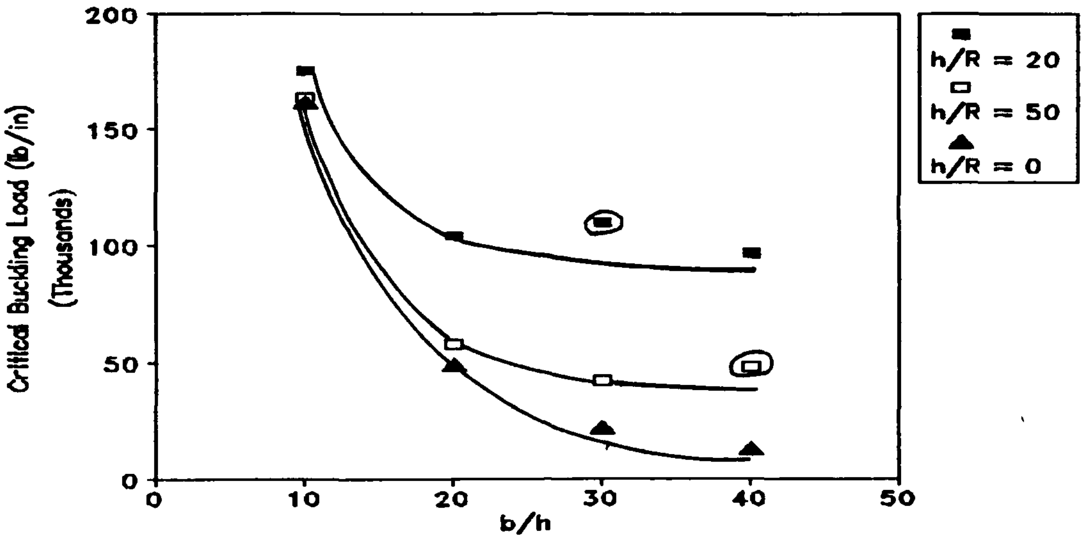

Figure 4. Curvature Effects on Critical Buckling Loads, Simply Supported Boundary, $[0/90]_s$

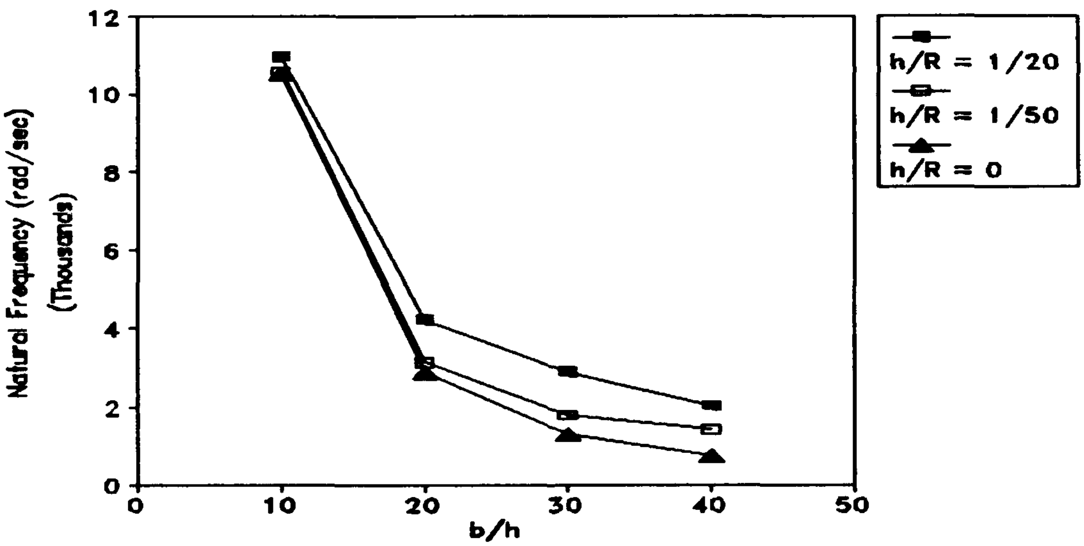

Figure 5. Curvature Effects for Natural Frequencies, Simply Supported Boundary, $[0/90]_s$

Compared to the $[0/90]_s$, the $[\pm 45]_s$ laminate does not converge immediately, however it does indicate excellent

convergence characteristics. Figures 6 and 7 show some typical results for both buckling loads and frequencies. Table 6 contains the corresponding numerical results. It demonstrates how the solutions decrease by smaller and smaller amounts with a corresponding increase M and N. It also indicates that the natural frequencies tend to converge more quickly than the buckling loads, thus requiring fewer terms in the assumed solution for convergence than a buckling problem.

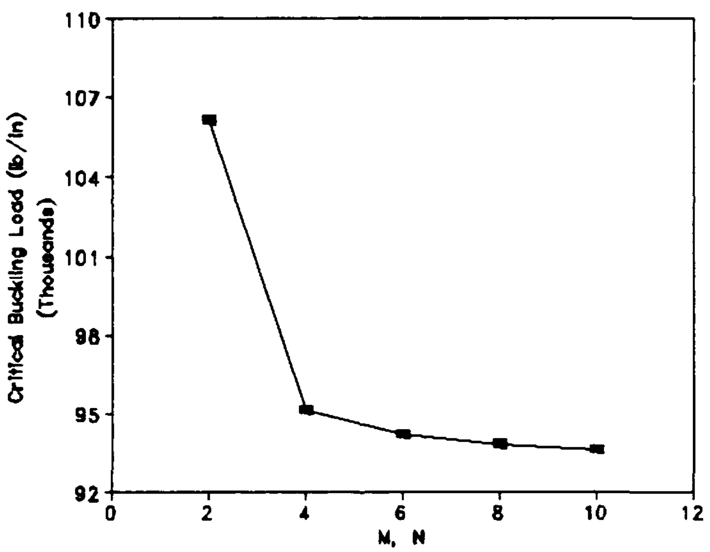

Figure 6. Convergence Characteristics for Critical Buckling Load, Simply Supported Boundary, $[\pm 45]_s$, a=b=20, h/R=1/20

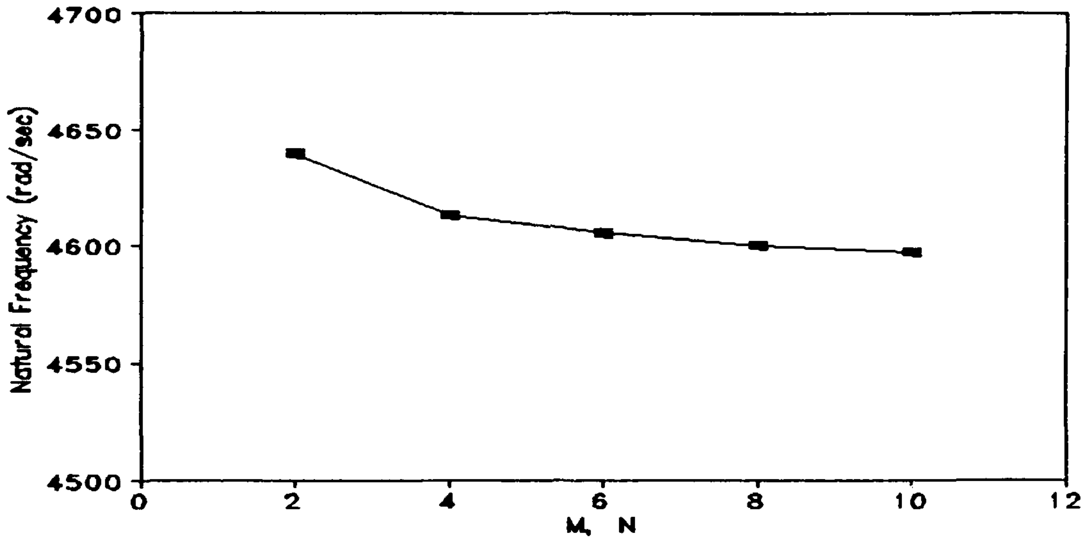

Figure 7. Convergence Characteristics for Natural Frequencies, Simply Supported Boundary, $[\pm 45]_s$, a=b=20, h/R=1/20

Table 6. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[\pm 45]_s$, Simply Supported (R = 50.0 in, h = 1.0 in, a = b = 20.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	106146.82	---	4639.26	---
4	95149.61	-1.036	4613.39	-0.56
6	94220.03	-0.98	4604.94	-0.19
8	93851.67	-0.39	4600.49	-0.10
10	93633.94	-0.23	4597.52	-0.06

The same type of convergence behaviors indicated above holds for all of the simply supported $[\pm 45]_s$ laminates.

Figures 8 and 9 show the same effects of curvature on the buckling loads and frequencies for the $[\pm 45]_s$ laminate as for the $[0/90]_s$ laminate. The loads and frequencies decrease with increasing span. Deeper panels are less affected than shallow shells or flat plates, and the frequencies are less affected than the buckling loads.

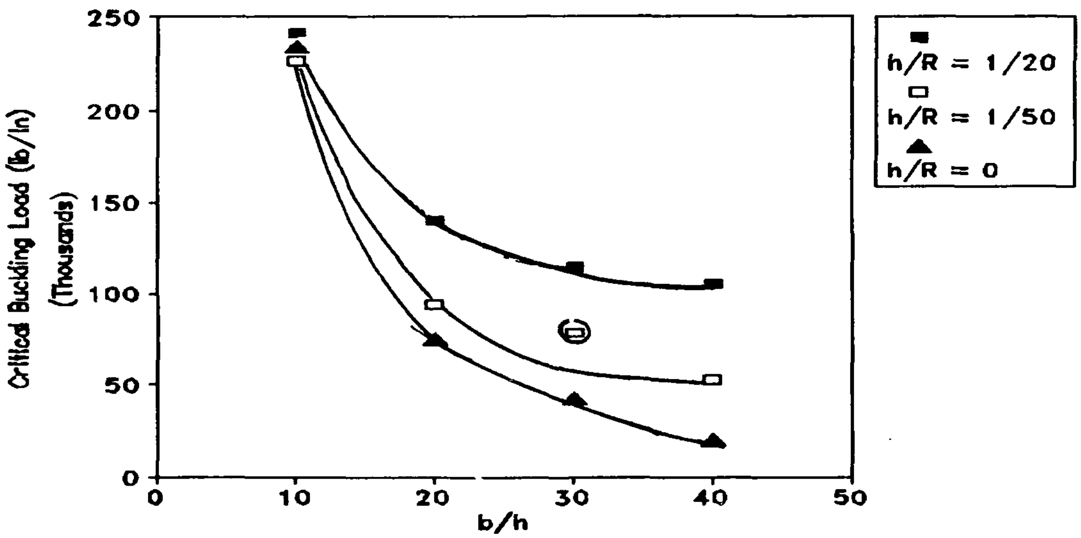

Figure 8. Curvature Effects for Critical Buckling Loads, Simply Supported Boundary, $[\pm 45]_s$

Up to this point, these results are not new. Linneman (10) examined symmetrical laminates using the same higher order shear theory applied herein. Validation of the results from this analysis was obtained from comparison to his

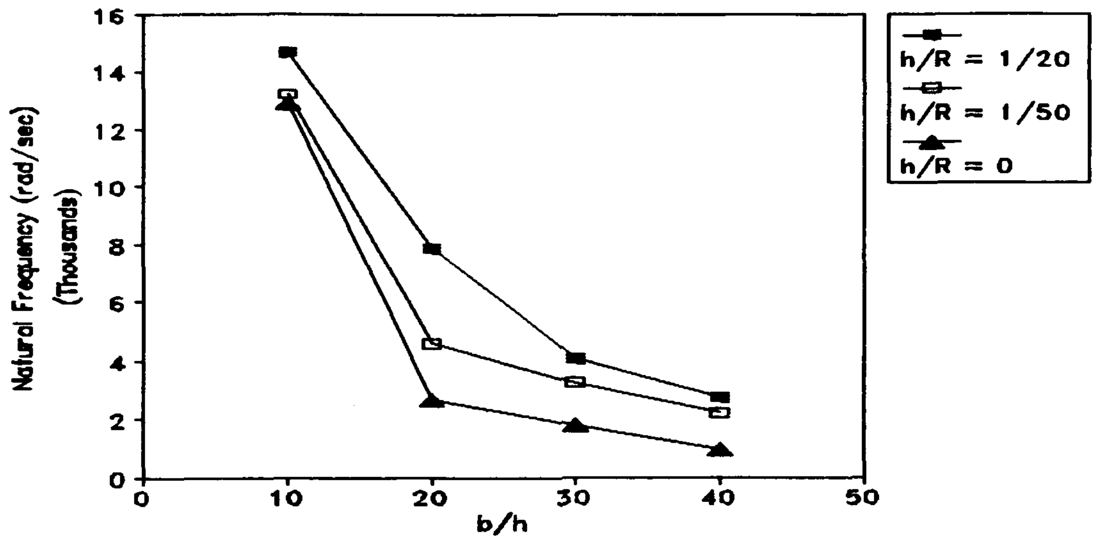

Figure 9. Curvature Effects for Natural Frequencies, Simply Supported Boundary, $[\pm 45]_s$

results. As part of his thesis, he made comparisons for the $[0/90]_s$ laminate with other theories. It is not the intent herein to duplicate his results, therefore a short summary of his important conclusions will be presented only to support the results presented in this work.

Reddy and Lui applied the Donnell theory, including parabolic transverse shear, to laminated circular cylindrical shell panels (21). They found an exact solution for simply supported boundary conditions, limited to $[0/90]_s$ laminates. Linneman showed that for large h/r values, (approaching a flat plate), the two methods compared very well. This is due to the fact that as h/r increases, the higher order terms in Sander's equations approach zero, reducing to Donnell's equations (4).

Comparisons to Jones' closed form solutions using Classical thin plate theory and to solutions from Bowlus and Palardy implementing Mindlin plate theory were also made (2; 9; 15). Classical theory, also known as the Kirchhoff-Love theory, assumes no transverse shear. This type of model produces a plate that is too stiff, and the buckling loads and frequencies resulting from this approach are too high. The Mindlin model uses a linear shear distribution. The parabolic shear distribution assumed in this work should produce lower frequencies than both of these theories. Linneman's comparisons verified that the parabolic results were indeed lower for both cases. The Mindlin solutions were very close for the $[0/90]_s$ simply supported boundary, due to the fact than an exact solution exists. For large a/h ratios of 40 to 50, the results approached the classical solution asymptotically. Linneman also looked at $[\pm 45]_s$ laminates with both simply supported and clamped boundaries. Again, the parabolic model produced lower frequencies. The results showed greater differences between the theories for the clamped results, as one would expect from the more complicated boundary.

To reiterate once more, Linneman performed these comparisons in his thesis, and they are included here only for the sake of completeness and to validate the results for the symmetric cases presented here. Anyone wishing further information is encouraged to review his work.

The intent of this thesis is to broaden the scope of his research and apply the theory to the additional cases described earlier: an additional layup of [0/±45/90] plies, the clamped-simple boundary condition, and non-symmetric laminates.

Results for the $[0/\pm45/90]_s$ laminates with simply supported boundaries also exhibited excellent convergence characteristics. Figures 10 and 11 show some typical results for both buckling loads and frequencies.

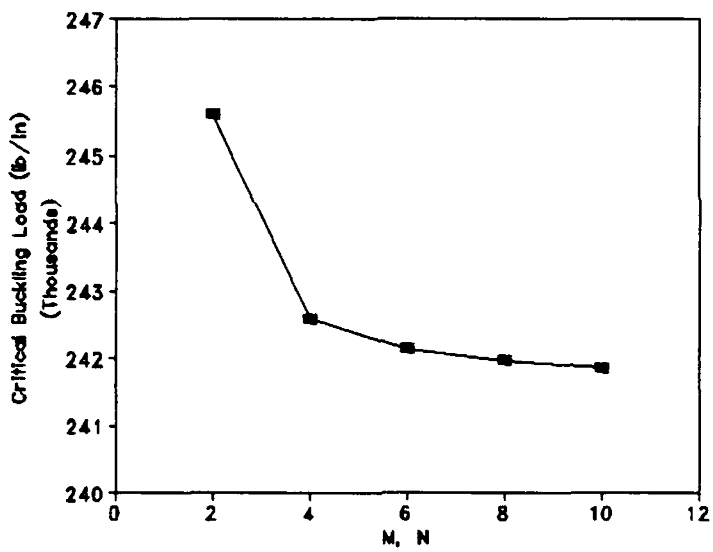

Figure 10. Convergence Characteristics for Critical Buckling Load, Simply Supported Boundary, $[0/\pm45/90]_s$, a=b=20, h/R=1/20

The same trends described in the previous ply lay-ups are apparent for this layup as well. The solutions show excellent

convergence characteristics, with the frequencies again converging more quickly than the buckling loads. The results also seem to converge more quickly than for the $[\pm 45]_s$ laminate, though it is not an immediate convergence as in the $[0/90]_s$ case. Figures 12 and 13 show the curvature effects for this laminate, with the same trends as found for the previous laminates. Again, similar to the $[0/90]_s$ laminate, there are points that seem inconsistent.

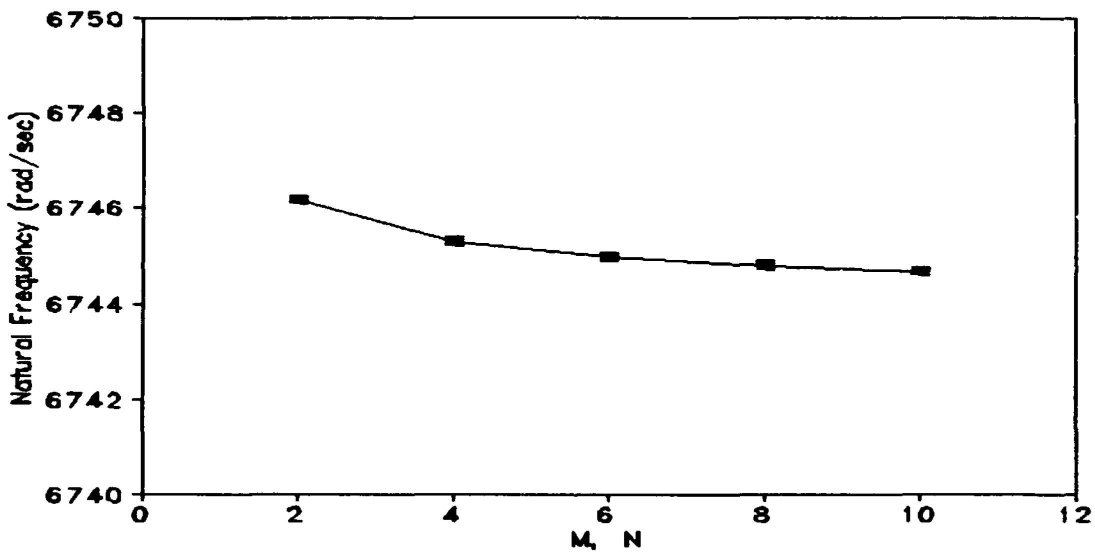

Figure 11. Convergence Characteristics for Natural Frequencies, Simply Supported Boundary, $[0/\pm 45/90]_s$, a=b=20, h/R=1/20

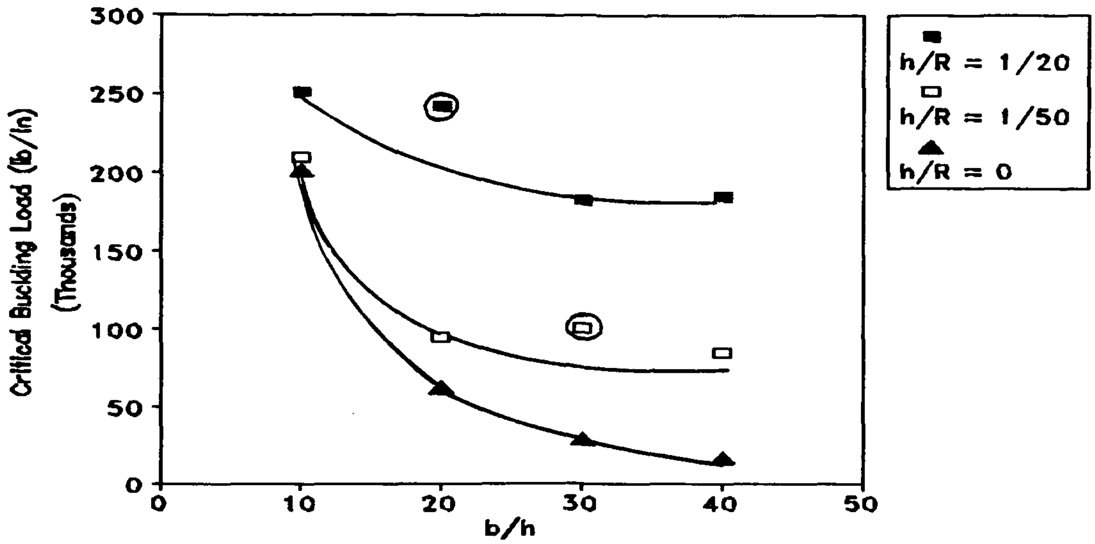

Figure 12. Curvature Effects for Critical Buckling Load, Simply Supported Boundary, $[0/\pm45/90]_s$

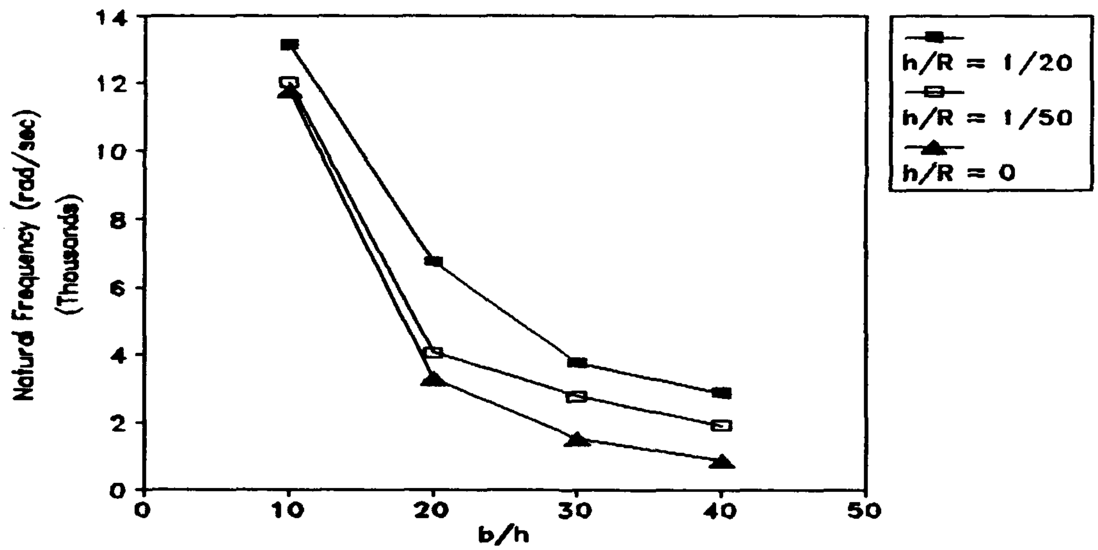

Figure 13. Curvature Effects for Natural Frequencies, Simply Supported Boundary, $[0/\pm45/90]_s$

From the comparisons of the stiffness terms for this ply layup against the corresponding stiffness terms for the $[0/90]_s$ and the $[\pm45]$ layups, one would expect the results to also fall between them. Figures 14 through 16 give graphical

comparisons. It can be seen that the magnitudes of the buckling loads and frequencies for the $[0/\pm45/90]_s$ layup are consistently higher than the loads and frequencies for the $[0/90]_s$ layup. This effect seems to decrease as h/r decreases, but increases with increasing a/h ratios. For a/h = 10, the buckling load is about 40% higher for h/R = 20, while at h/R = 0, the difference is only about 25% higher. At a/h = 40, the difference decreases from about 90% at h/r = 20 to 30% at h/R = 0.

The frequencies follow the same trends, except that the per cent differences are smaller. For a/h = 10, the frequencies for the $[0/\pm45/90]_s$ range from about 20% higher at h/R = 20 to 10% higher at h/R = 0. For b/h = 40, the frequencies range from about 40% higher at h/R = 20 to about 10% higher at h/R = 0.

Comparison was next made with the $[\pm45]_s$ laminate. Trends were not as obvious in this case. The frequencies resulting from this case were consistently lower than the $[\pm45]_s$ case, with only one exception. There did not seem to be an obvious relationship between differences and the dimensions of the panel, as there seemed to be in comparison with the $[0/90]_s$ layup.

Comparison of the buckling loads for the two ply layups produced very different results than for the frequencies. The buckling loads were not consistently lower as were the

frequencies. For a curvature of h/R of 1/20, the buckling loads were in fact larger for the $[0/\pm45/90]_s$ laminate than for the $[\pm45_s]$ laminate (see Figure 14). However, as the radius increased, the loads did become less than those for the $[\pm45]_s$ laminate. Figure 15 shows the same plot for h/R of 1/50. For larger panels, b/h of 30 and 40, only for h/R equal to zero did the loads become smaller.

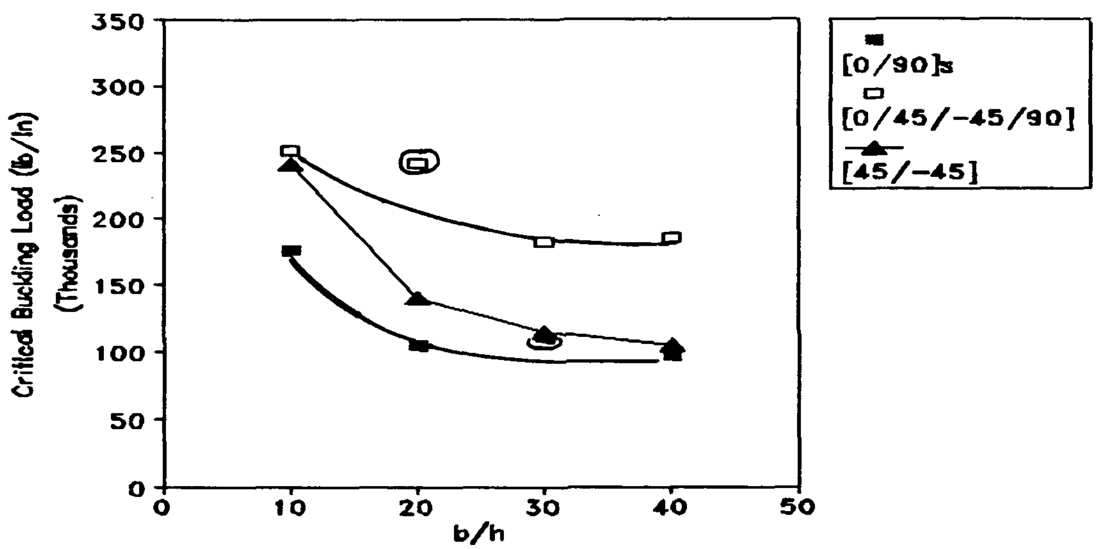

Figure 14. Comparison of Critical Buckling Loads for Different Ply Layups, Simply Supported Boundary, h/R=1/20

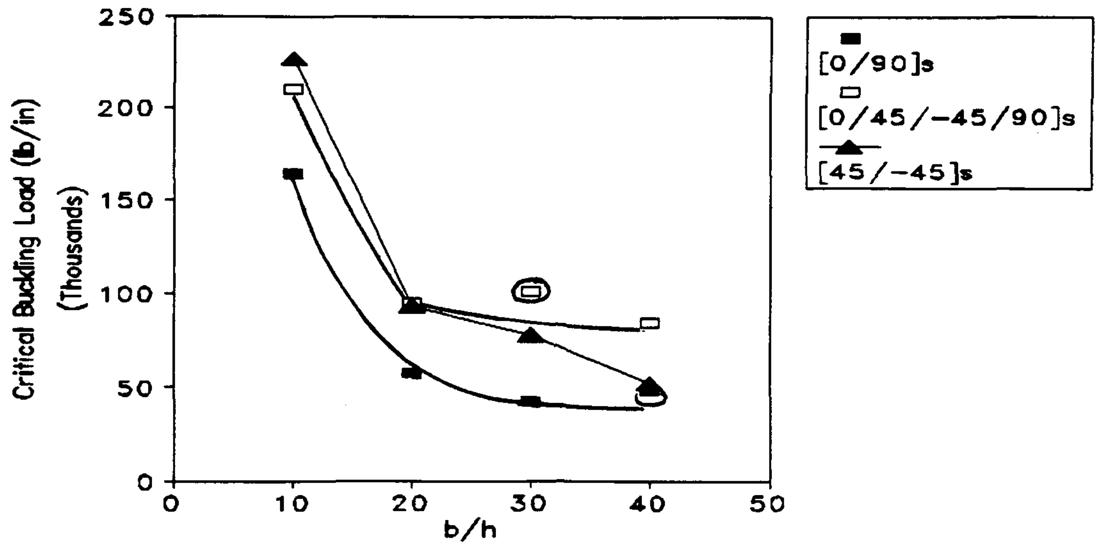

Figure 15. Comparison of Critical Buckling Loads for Different Ply Layups, Simply Supported Boundary, h/R=1/50

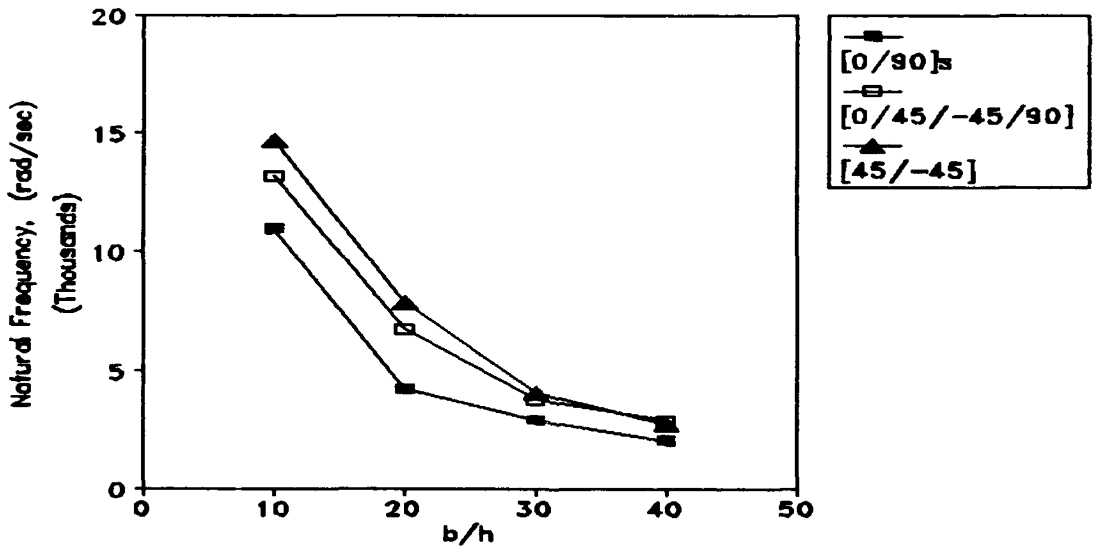

Figure 16. Comparison of Natural Frequencies for Different Ply Layups, Simply Supported Boundary, h/R=1/20

The effect of nonsymmetric laminates on the buckling loads and natural frequencies was studied next. A nonsymmetric laminate will have nonzero terms in its B_{ij} stiffness matrix, which represents the coupling between extension and bending. The same holds true for the corresponding higher order stiffness matrices $[E_{ij}]$, $[G_{ij}]$, and $[I_{ij}]$. The effect of this coupling should make the laminate less stiff, and thus decrease the buckling loads and frequencies. Jones presented solutions for which the effect of nonsymmetry lowered frequencies by as much as 40% (9:248-283). Whitney also preformed work in this area, and produces similar results (28).

First, a [0/90/0/90] ply layup was assumed, (or $[0/90]_{as}$), and entered into the simply supported problem. Its stiffness elements are presented in Table 4. For this particular laminate, the solutions are found to converge immediately at M - N = 2. Again, an exact solution exists.

The results for both the critical buckling loads and the natural frequencies follow the same trends found for the symmetric case. The effects of varying the curvature and span to thickness ratio is the same as observed for the symmetric laminates.

The results when compared to the symmetric case are not quite as expected. Figures 17 through 20 show graphical comparisons. For both cases, the frequencies and buckling loads are very close. In nearly all cases, the results for

the antisymmetric case were less than those of the symmetric case. Yet for large values of a/h, some increases were seen. This is not as the theory governing laminate behavior would predict.

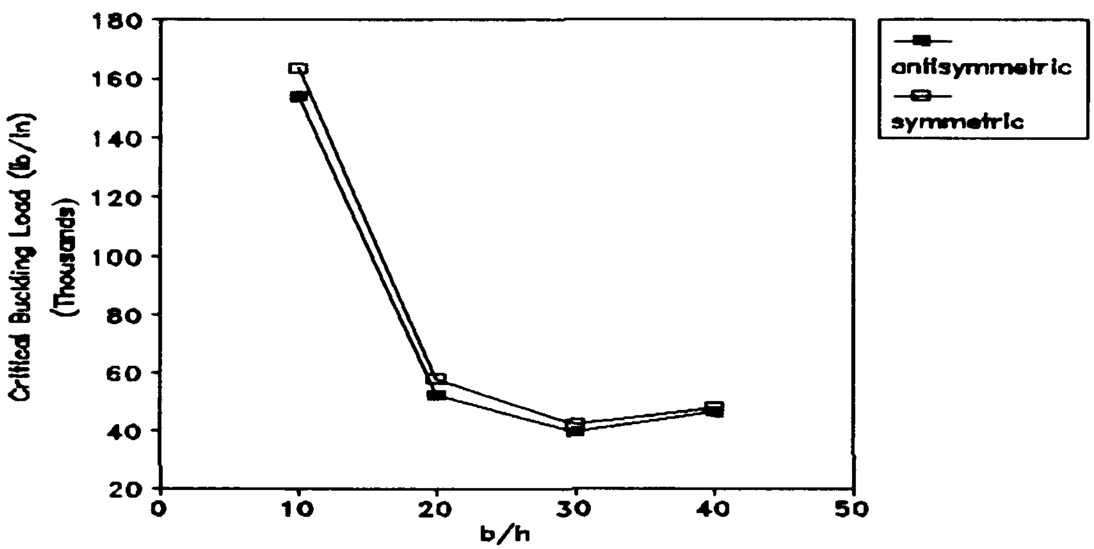

Figure 17. Comparison of Critical Buckling Loads for $[0/90]_s$ vs $[0/90]_{as}$ Simply Supported Boundary, h/R=1/50

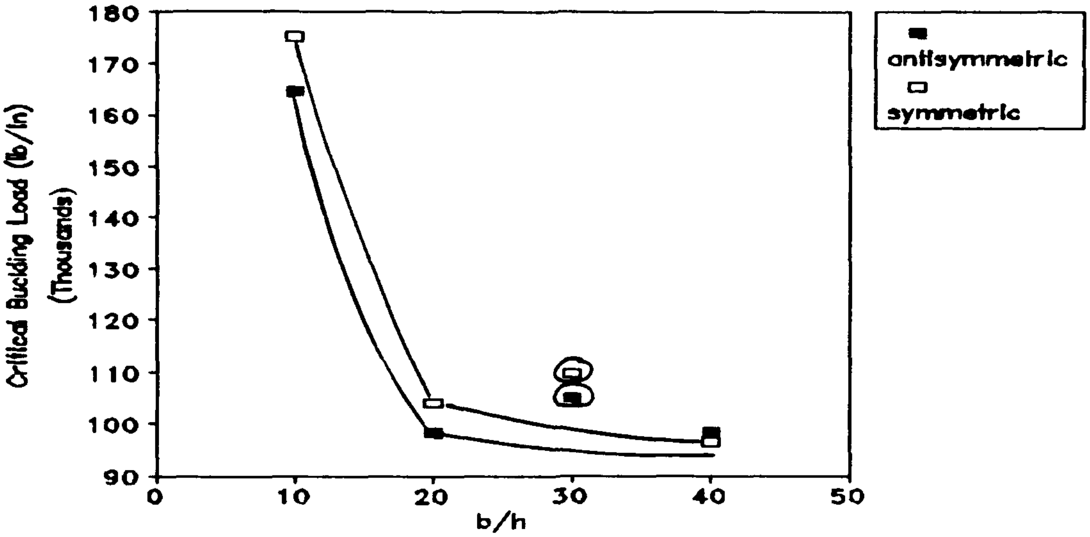

Figure 18. Comparison of Critical Buckling Loads for $[0/90]_s$ vs $[0/90]_{as}$ Simply Supported Boundary, h/R=1/20

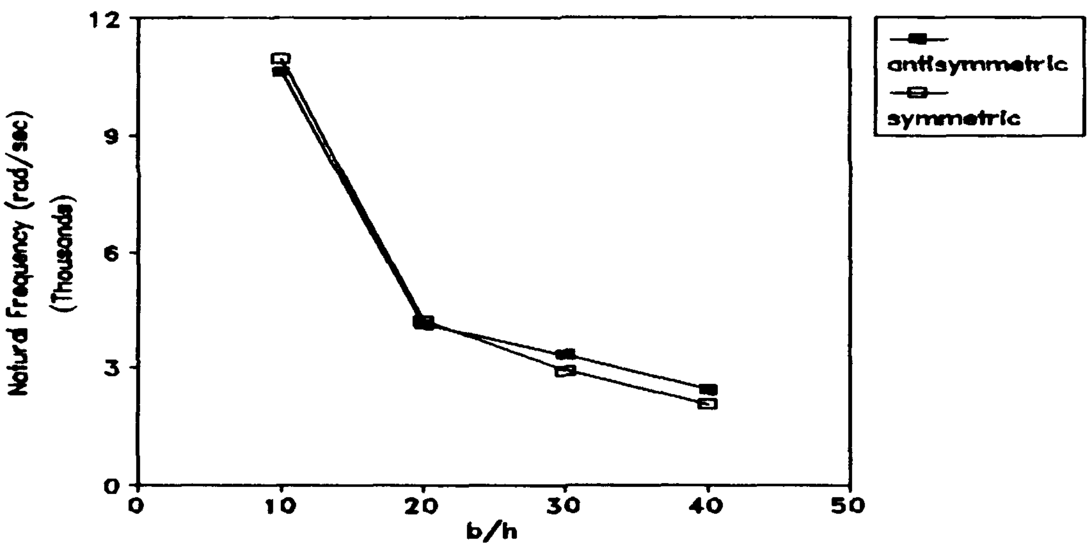

Figure 19. Comparison of Natural Frequencies for $[0/90]_s$ vs $[0/90]_{as}$ Simply Supported Boundary, h/R=1/20

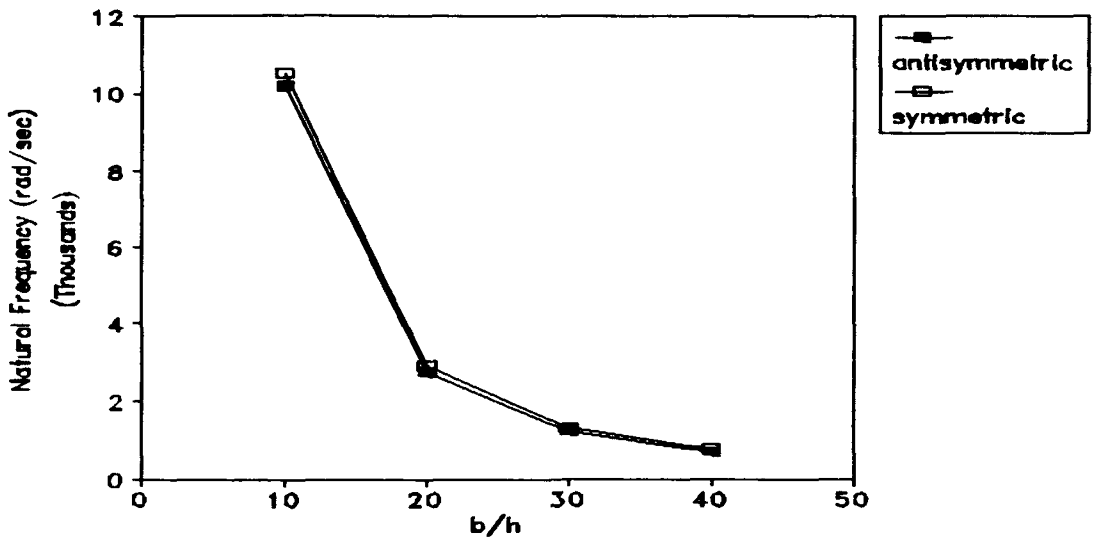

Figure 20. Comparison of Natural Frequencies for $[0/90]_s$ vs $[0/90]_{as}$ Simply Supported Boundary, h/R=0

Next, a $[\pm 45]_{as}$ ply layup was investigated. Its stiffness elements are presented in Figure 5. Results for this laminate were similar to the results found for the $[0/90]_{as}$ layup. Figures 21 through 24 show graphical comparisons for buckling loads and natural frequencies. In Figure 21, the buckling loads for the antisymmetric laminate are higher than the symmetric laminate, on the order of 10% for an h/R value of 1/20. Increasing the radius resulted in loads that were closer to the symmetric case, as shown in Figure 22, yet still higher. Figures 22 and 23 show the natural frequencies for the antisymmetric laminate were typically very close to the frequencies for the symmetric case, but again slightly higher.

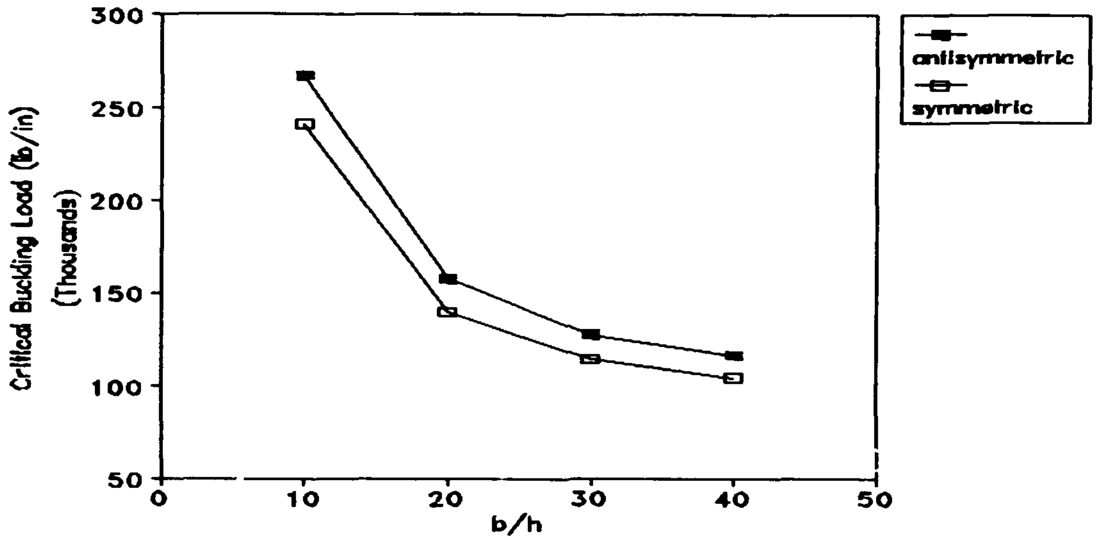

Figure 21. Comparison of Critical Buckling Loads for $[\pm 45]_s$ vs $[\pm 45]_{as}$ Simply Supported Boundary, h/R = 1/20

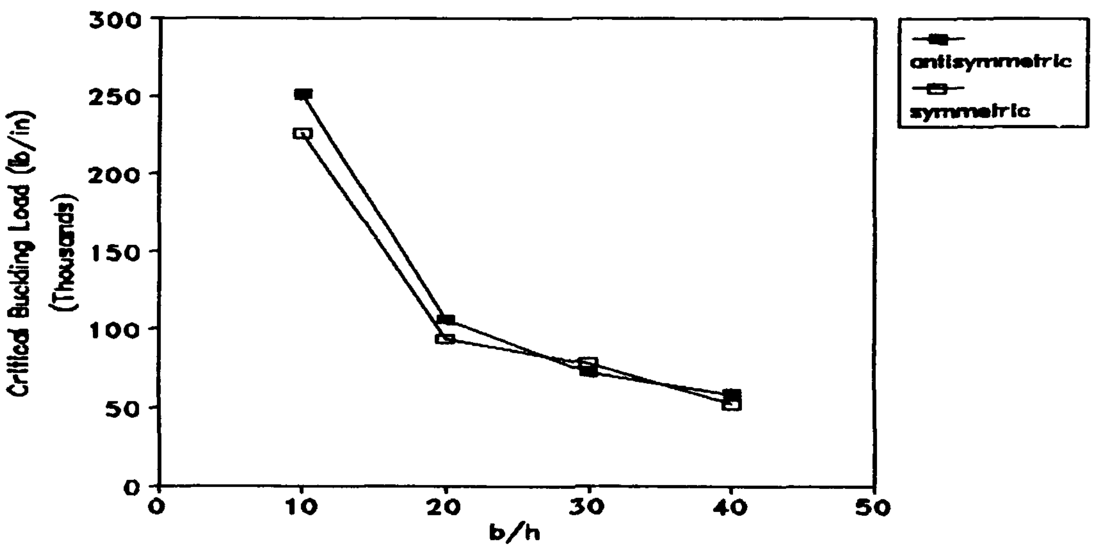

Figure 22. Comparison of Critical Buckling Loads for $[\pm 45]_s$ vs $[\pm 45]_{as}$ Simply Supported Boundary, h/R=1/50

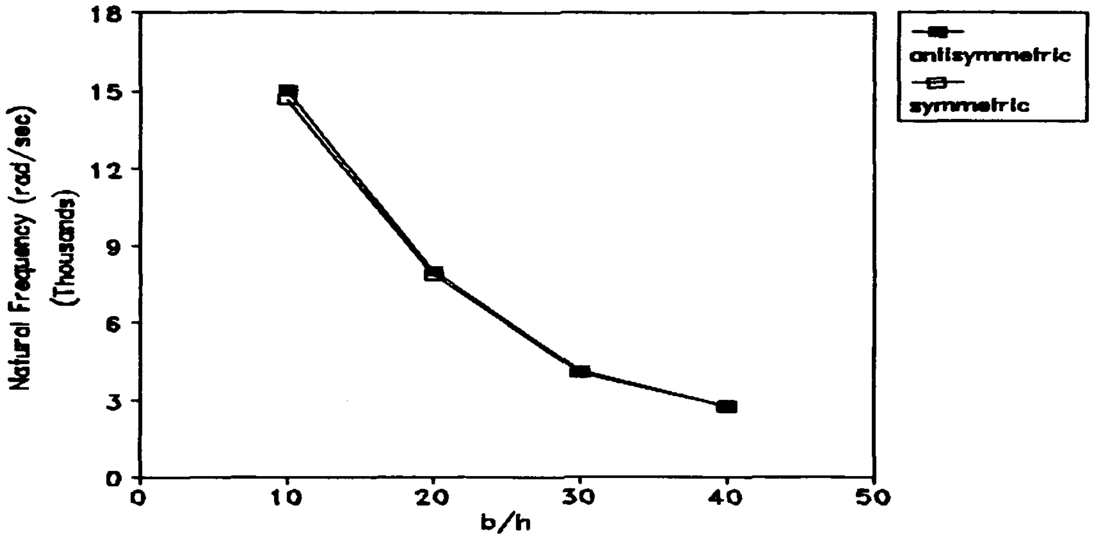

Figure 23. Comparison of Natural Frequencies for $[\pm45]_s$ vs $[\pm45]_{as}$ Simply Supported Boundary, h/R=1/20

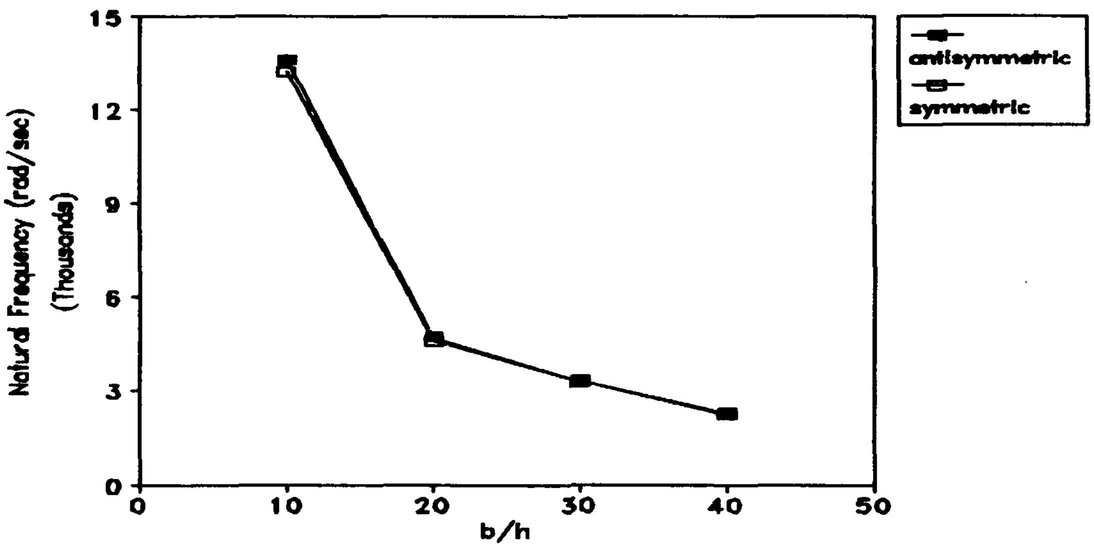

Figure 24. Comparison of Natural Frequencies for $[\pm45]_s$ vs $[\pm45]_{as}$ Simply Supported Boundary, h/R=1/50

Several hypotheses were investigated to determine possible causes for these results. No errors were found in the FORTRAN code after extensive searching. In addition, the excellent agreement with Linneman's results, plus the logical and consistent trends established and followed for all cases investigated would seem to indicate no programming errors.

The behavior of the general eigenvalue problem was also investigated. It was thought that the matrices built from the Galerkin equations might be ill-conditioned, or very sensitive to small perturbations (6:598-604; 30:941-949). This also turned out not to be the case.

Two other ideas were proposed that may lend some insight on these unexpected results. Jones (9:255-283) presents result for antisymmetric laminated flat plates, having high modulus ratios, E_1/E_2 = 40 and ply thickness of 0.005 inch. He found that for large numbers of plies, the effect of bending-coupling decreases rapidly. Whitney (28) made similar conclusions. The laminates investigated in this work all have thicknesses of 1.0 inch. These would have the equivalent of 200 plies in Jones' analysis. Reddy (22) also found that material with lower modulus ratios were less effected by nonsymmetry.

It may be possible that the relatively large thickness of the plies, or the moderately low modulus ratio for the

material used in this analysis, $E_1/E_2 = 15$, is counteracting the effect of the nonsymmetry.

Additionally, Jones describes cases where similar results were found, using the Rayleigh-Ritz method, due to a relatively inaccurate solution and the inability to satisfy natural boundary conditions (9:282). Chen and Shu (5) alluded to finding similar results for some cases using a large deflection model. These cases may or may not be related to the cause of the unexpected results found in this thesis, yet it does indicate that the results obtained here are not unique.

Finally, the effects of the higher order stiffness terms, F_{ij}, G_{ij}, etc., are also unknown. The effects of the A_{ij}, B_{ij}, and D_{ij} stiffness elements have been documented by many investigators. These higher order elements however, are not fully understood. It may be possible that the presence of these terms in a nonsymmetric laminate cause unexpected behavior.

Clamped Boundary Condition

The next case investigated is the clamped boundary condition. The results for this case were examined for the same trends as were the simply supported cases. First, symmetric layups were run, and compared to results obtained from Linneman (10). The results compare very well, again serving to validate the work performed in this analysis. Following this validation, the effects of the $[0/\pm45/90]_s$ ply layup and nonsymmetry are investigated for this boundary.

The solutions are first checked for convergence, as described previously. The assumed solutions do not satisfy the natural clamped boundary conditions, therefore the resulting buckling loads and frequencies are not expected to converge as quickly as for the simply supported case.

Figures 25 and 26 show some typical convergence characteristics for the $[0/90]_s$ laminate. The results verify that the convergence is not as fast as for the simply supported case, for either problem, but indicate good convergence characteristics nonetheless. Numerical results presented in Tables 7 and 8 demonstrate that the frequencies converge more rapidly than the buckling loads, as in the simply supported case.

An interesting note is the convergence behavior for short (b/h = 10) panels. This particular laminate behaves differ-

ently in convergence than the others. While the larger laminates converge from a large value and decrease by smaller and smaller amounts, this case does not. As indicated in Figure 25, there is an initial decrease, then the solutions are seen to increase by smaller amounts.

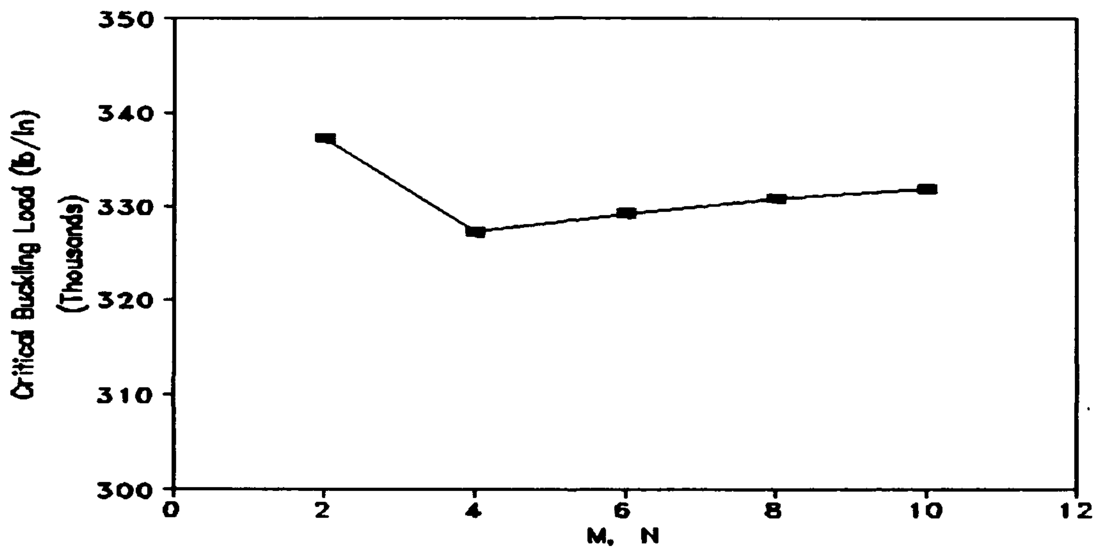

Figure 25. Convergence Characteristics for Critical Buckling Loads, Clamped Boundary $[0/90]_s$, a=b=10, h/R=1/20

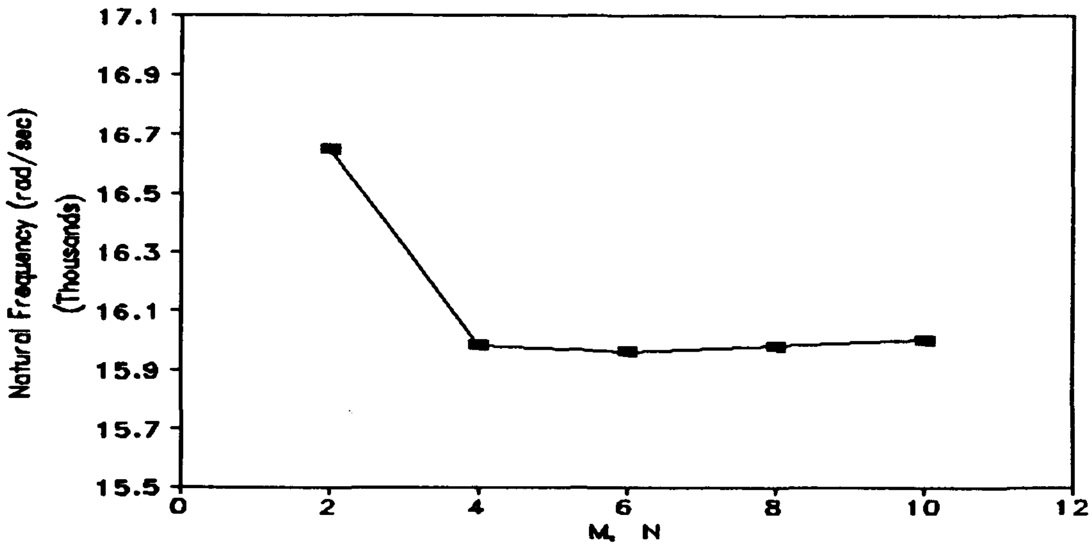

Figure 26. Convergence Characteristics for Natural Frequencies, Clamped Boundary, $[0/90]_s$, a=b=10, h/R=1/20

Table 7. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[0/90]_s$, Clamped (R = 20.0 in, h = 1.0 in, a = b = 10.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	337233.43	---	16649.12	---
4	327211.45	-2.97	15982.47	-4.00
6	329209.28	+0.61	15960.64	-0.14
8	330801.91	+0.48	15979.57	+0.12
10	331868.15	+0.32	16001.19	+0.14

Table 8. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[0/90]_s$, Clamped (R = 50.0 in, h = 1.0 in, a = b = 40.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	223823.22	---	3112.16	---
4	79166.73	-64.63	2092.27	-32.77
6	77013.96	-2.72	2023.20	-3.30
8	76327.19	-0.89	2003.47	-0.98
10	76048.48	-.36	1996.57	-0.34

Figures 27 and 28 show the effects of curvature for this laminate. Once again, the effects of deeper shells on the buckling loads and frequencies are the same. The buckling loads are greatly affected by the curvature, increasing considerably for an h/R ratio of 1/20 versus a flat plate. The frequencies are also seen to increase, as expected. Compared with the buckling load behavior, the effect is relatively small. This is just as seen for the simply supported boundary.

Compared with the results for the simply supported case, the buckling loads and frequencies for the clamped $[0/90]_s$ laminate are much larger for all configurations investigated. The difference is quite dramatic for the buckling loads, on the order of 50 to 90% larger for h/R of 1/20, and as h/R approaches zero, up to four times larger. The differences for

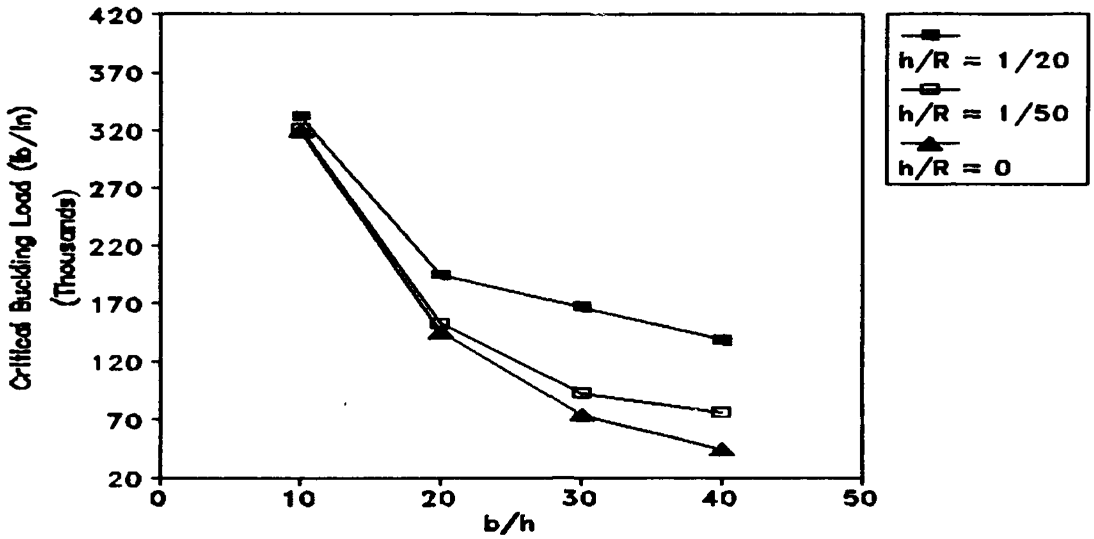

Figure 27. Curvature Effects on Critical Buckling Loads, Clamped Boundary, $[0/90]_s$

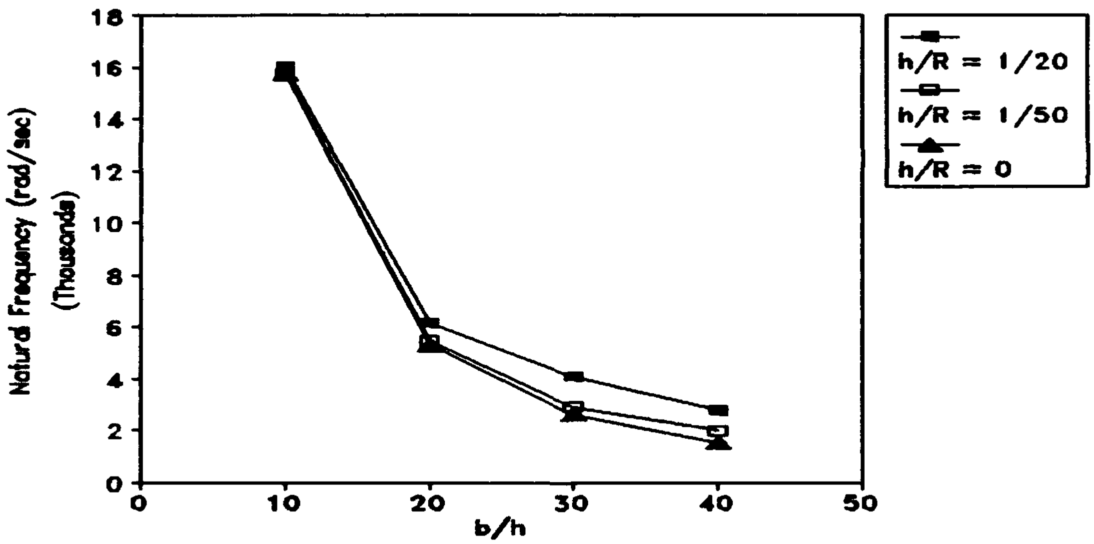

Figure 28. Curvature Effects on Natural Frequencies, Clamped Boundary, $[0/90]_s$

the frequencies are not quite so large, on the order of 40% for h/R of 1/20, but still reaching over 100% for flat plates.

The $[\pm 45]_s$ laminate also produced results that are larger than the corresponding layup in the simple supported boundary, however, not nearly as much as the $[0/90]_s$ case. The buckling loads and frequencies are on the order of 10 to 20% higher for large h/R values, and 20 to 50% higher as h/R approaches zero. The buckling loads follow a similar scale. Some results for the $[\pm 45]_s$ laminated are presented in Figures 29 and 30, and numerical results in Tables 9 and 10. Again, convergence appears very good. However, in some of the shorter laminates, b/h = 10, slight increases appear in the results for M = N = 10. These increases are very small,and could be caused by simple round-off errors. However, they may be indications of divergence beyond M = N = 10.

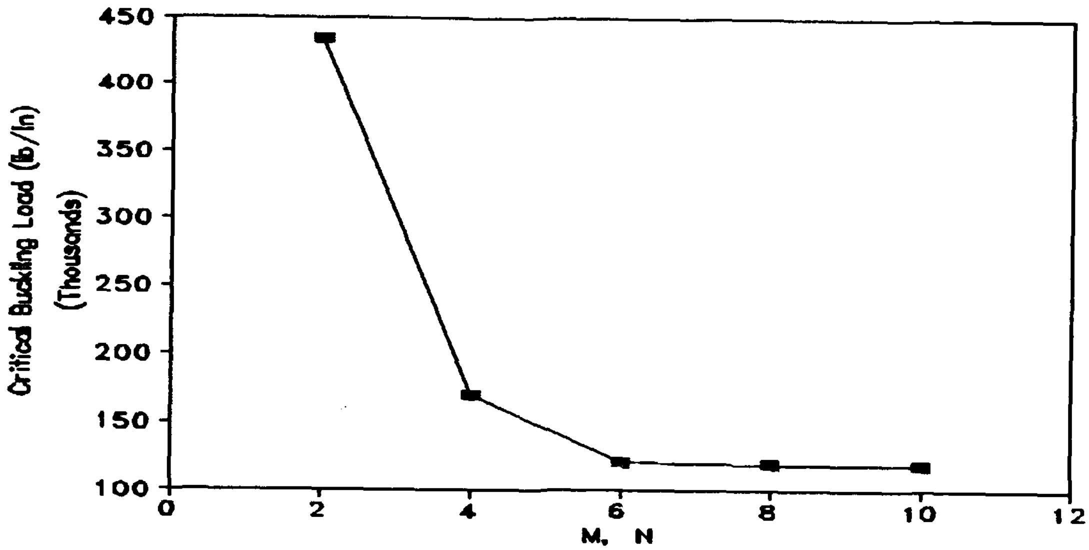

Figure 29. Convergence Characteristics for Critical Buckling Loads, Clamped Boundary, $[\pm 45]_s$, a=b=30, h/R=1/20

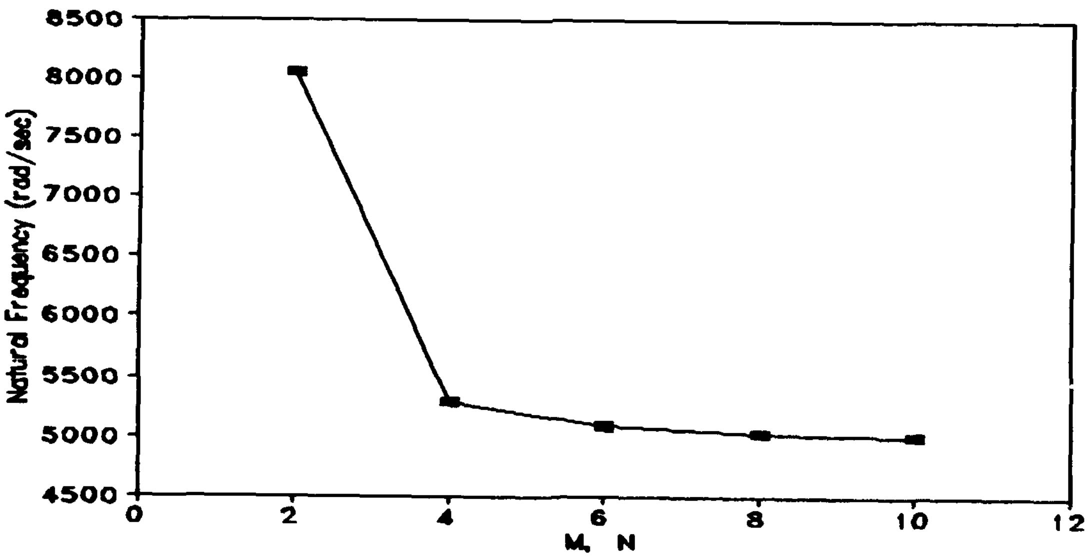

Figure 30. Convergence Characteristics for Natural Frequencies, Clamped Boundary, $[\pm 45]_s$, a=b=30, h/R=1/20

Table 9. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[\pm 45]_s$, Clamped (R = 50.0 in, h = 1.0 in, $a = b$ = 10.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	334998.03	---	16535.28	---
4	246979.45	-26.27	15817.05	-4.34
6	245613.00	-0.55	15752.59	-0.41
8	245481.93	-0.05	15752.74	---
10	2445549.11	+0.03	15762.25	+0.06

Table 10. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[\pm 45]_s$, Clamped (R = 20.0 in, h = 1.0 in, $a = b$ = 30.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	434764.02	---	8055.56	---
4	169984.12	-60.90	5298.00	-3.23
6	120581.26	-29.09	5099.41	-3.75
8	119528.51	-0.87	5036.63	-1.23
10	119299.60	-0.19	5011.82	-0.49

Figures 31 and 32 below show the curvature effects for this laminate, and as expected, it also exhibits the same behavior as all the previous cases.

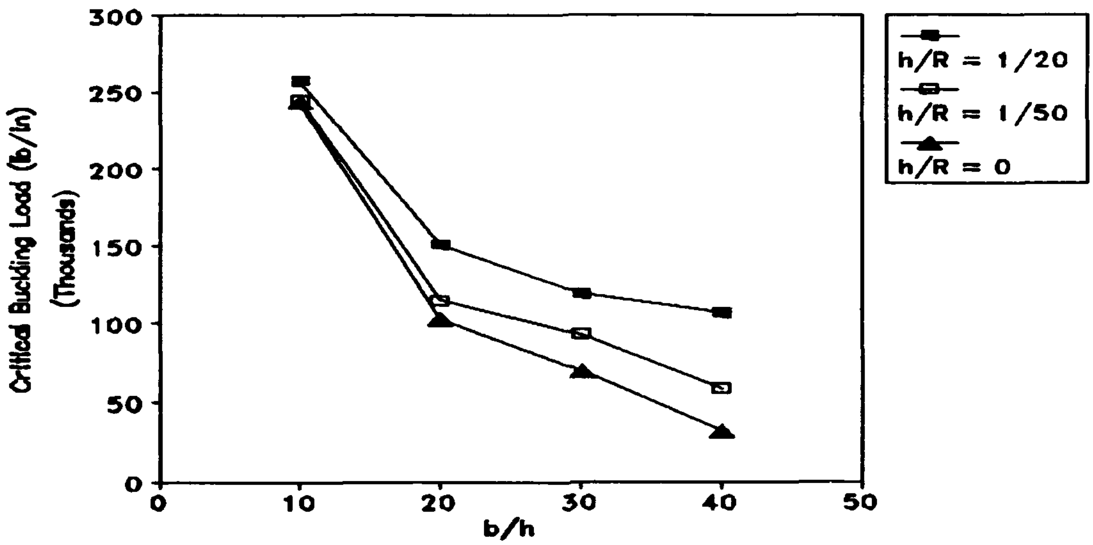

Figure 31. Curvature Effects on Critical Buckling Loads, Clamped Boundary, $[\pm45]_s$

The comparison of the $[0/90]_s$ and the $[\pm45]_s$ laminates for the clamped boundary produced some very interesting results. While both the buckling loads and natural frequencies were consistently lower for the $[0/90]_s$ simply supported laminate than for the $[\pm45]_s$ laminate, this is not the case for the clamped condition. The frequencies are lower for the $[\pm45]_s$ laminate for flat plates, and as the curvature increases, the frequencies become greater than those of the $[0/90]_s$ laminate. The buckling loads are consistently larger for the $[0/90]_s$

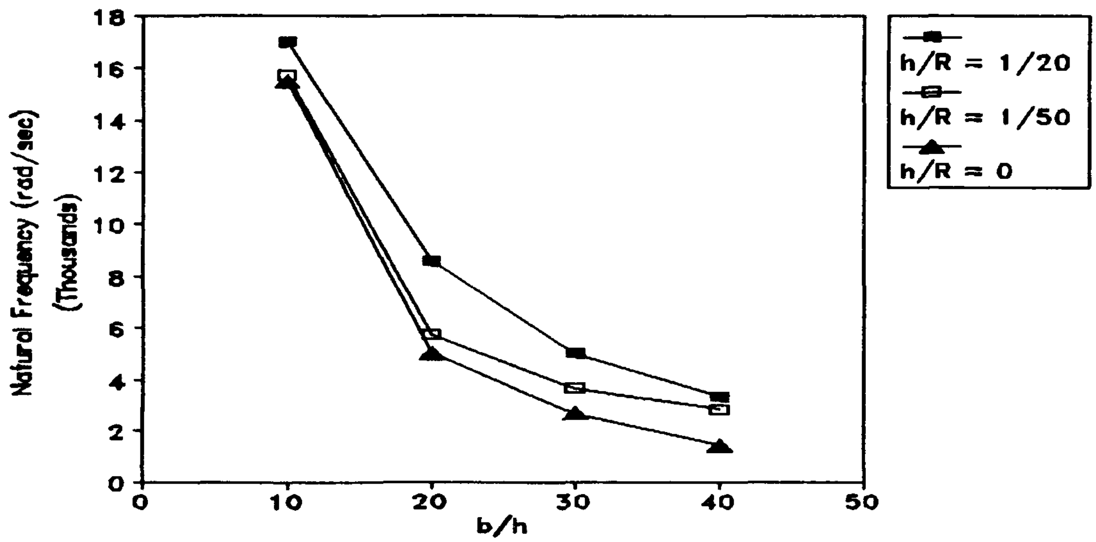

Figure 32. Curvature Effects on Natural Frequencies, Clamped Boundary, $[\pm 45]_s$

laminate.

This would initially appear to be an inconsistency. One would think that the buckling loads and natural frequencies would behave similarly. Looking at the results, there is no aberrant behavior that would indicate this an error; convergence characteristics are excellent, and all the results follow the trends established previously.

The $[0/\pm 45/90]_s$ ply layup was next investigated for the clamped boundary. Its convergence characteristics are similar to those of the $[\pm 45]_s$ laminate, in that for a/h = 10, the solutions display an initial decrease, and then increase to asymptotically approach a constant value. Some results are presented in Figures 33 and 34 below, with numerical results in Tables 11 and 12.

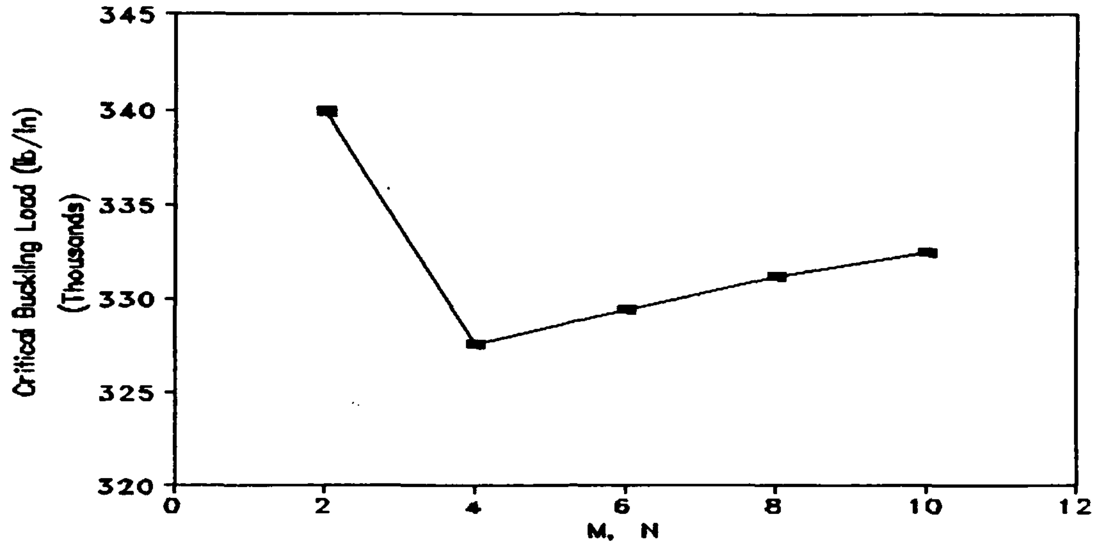

Figure 33. Convergence Characteristics for Critical Buckling Loads, Clamped Boundary, $[0/\pm45/90]_s$, a=b=10, h/R=1/50

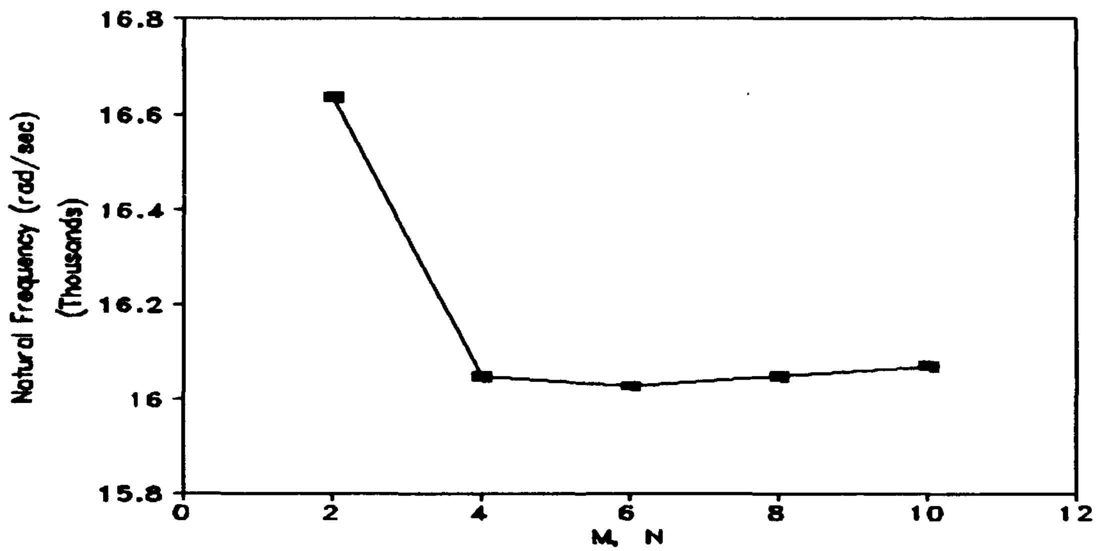

Figure 34. Convergence Characteristics for Natural Frequencies, Clamped Boundary, $[0/\pm45/90]_s$, a=b=10, h/R=1/50

Table 11. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[0/\pm45/90]_s$, Clamped (R = 50.0 in, h = 1.0 in, a = b = 10.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	339951.68	---	16635.43	---
4	327579.00	-3.64	16047.65	-3.53
6	329423.18	+0.55	16027.12	-0.13
8	331201.89	+0.54	16046.70	+0.12
10	332469.38	+0.38	16068.69	+0.14

Table 12. Convergence of Critical Buckling Load (lb/in) and Natural Frequency (rad/sec) for $[0/\pm45/90]_s$, Clamped, (R = 20.0 in, h = 1.0 in, a = b = 20.0 in)

M = N	$\bar{N}_1$	% Decrease	ω	% Decrease
2	402111.19	---	8839.18	---
4	267400.17	-33.50	7993.68	-9.57
6	265808.95	-0.60	7937.18	-0.70
8	265534.83	-0.10	7927.97	-0.12
10	265499.61	-0.01	7928.33	-0.01

Curvature effects are the same as previous cases, deeper shells showing increases in frequency and even larger increases in buckling loads. These results are presented in Figures 35 and 36 below. These indicate once again a small increase in the results for the natural frequencies at $M = N = 10$.

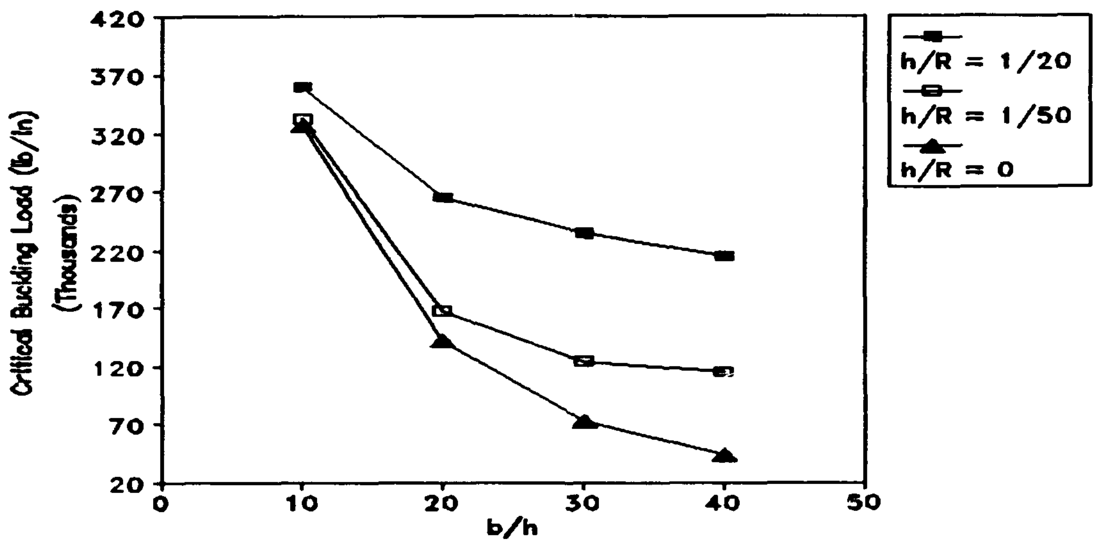

Figure 35. Curvature Effects on Critical Buckling Loads, Clamped Boundary, $[0/\pm45/90]_s$

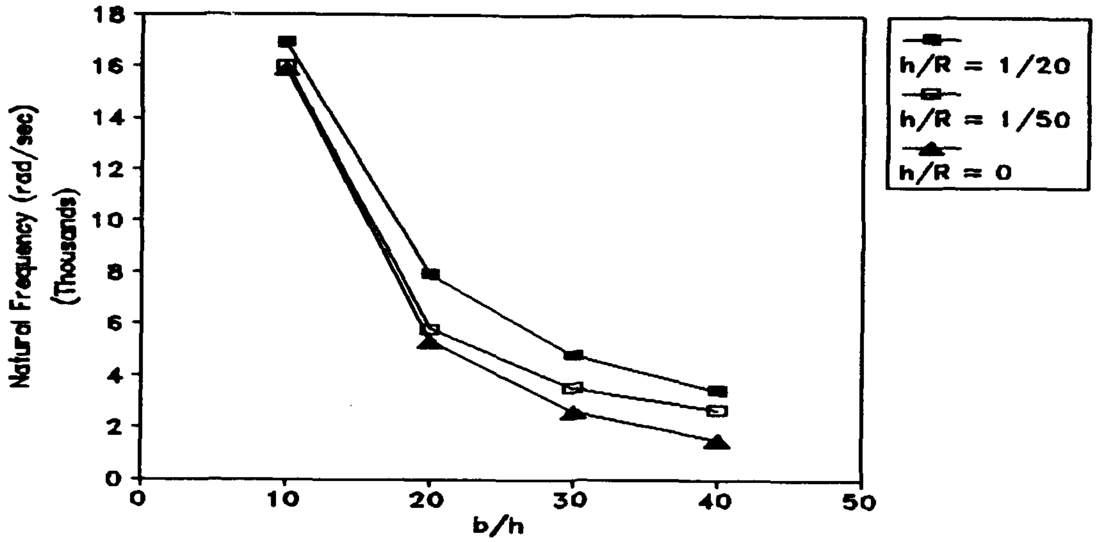

Figure 36. Curvature Effects on Natural Frequencies, Clamped Boundary, [0/±45/90]

The results were then compared to the $[0/90]_s$ and $[[\pm 45]_s$ laminates, as was done for the simply supported case. Some of these results are shown in Figures 37 and 38. Some comparable trends were found: the buckling loads and frequencies for the $[0/\pm 45/90]_s$ laminate are again much higher than those for the $[0/90]_s$.

In comparison to the $[\pm 45]_s$ laminates, the buckling loads are consistently higher for the $[0/\pm 45/90]_s$ laminates. The loads are on the order of 40% higher for small a/h ratios and increase to approximately 100% higher for a/h of 40. The differences between the loads are also seen to decrease with increasing radius.

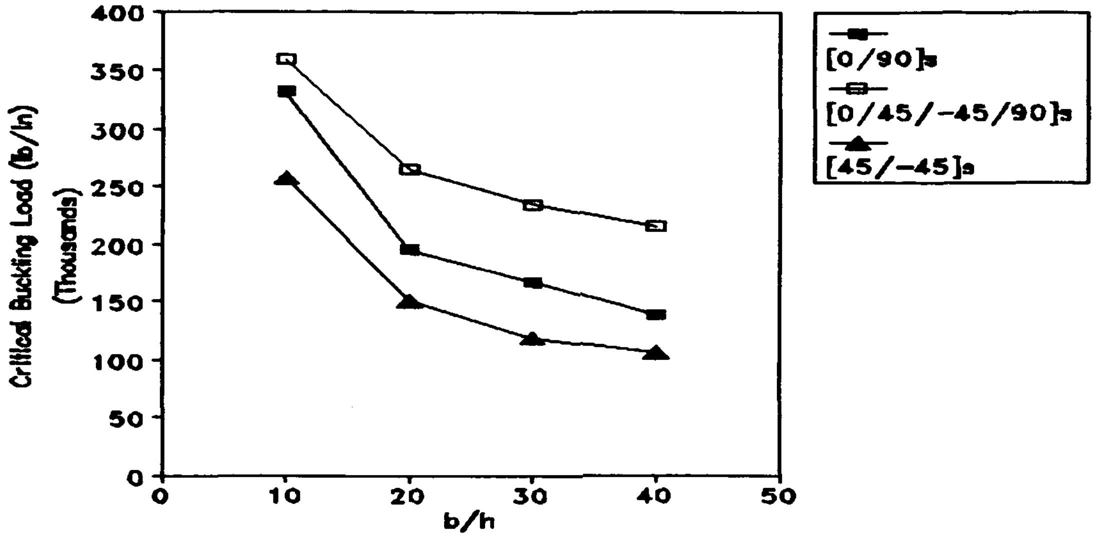

Figure 37. Comparison of Critical Buckling Loads for Different Ply Layups, Clamped Boundary, h/R=1/20

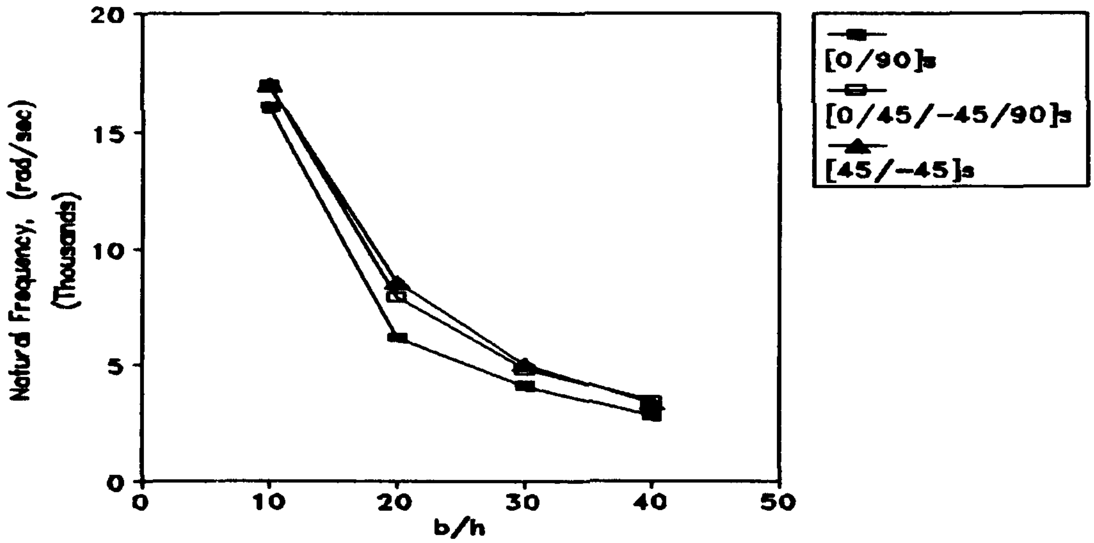

Figure 38. Comparison of Natural Frequencies for Different Ply Layups, Clamped Boundary, h/R=1/20

However, the behavior of the frequencies is not comparable to the simply supported case, where the frequencies were

lower for the [0/±45/90] laminates. Rather, for the clamped boundary, the frequencies are larger than the $[\pm 45]_s$ laminates for about half of the cases. There did not seem to be an obvious trend.

The effect of antisymmetric ply layup were also investigated for the clamped boundary condition. Some results are shown below including comparable results for the $[0/90]_s$ laminates, Figure 39 through Figure 42.

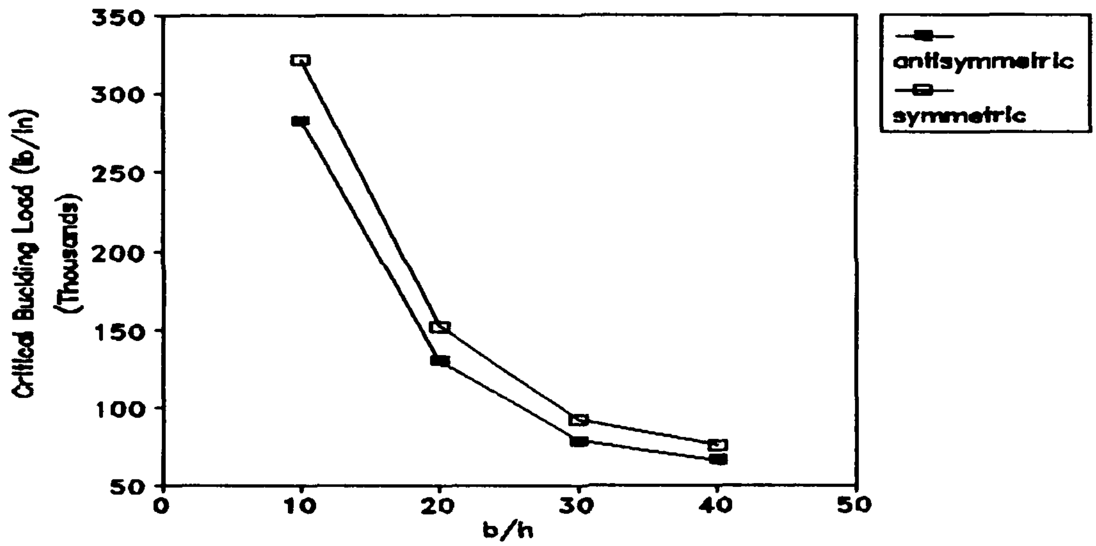

Figure 39. Comparison of Critical Buckling Loads for $[0/90]_s$ vs $[0/90]_{as}$ Clamped Boundary, h/R=1/50

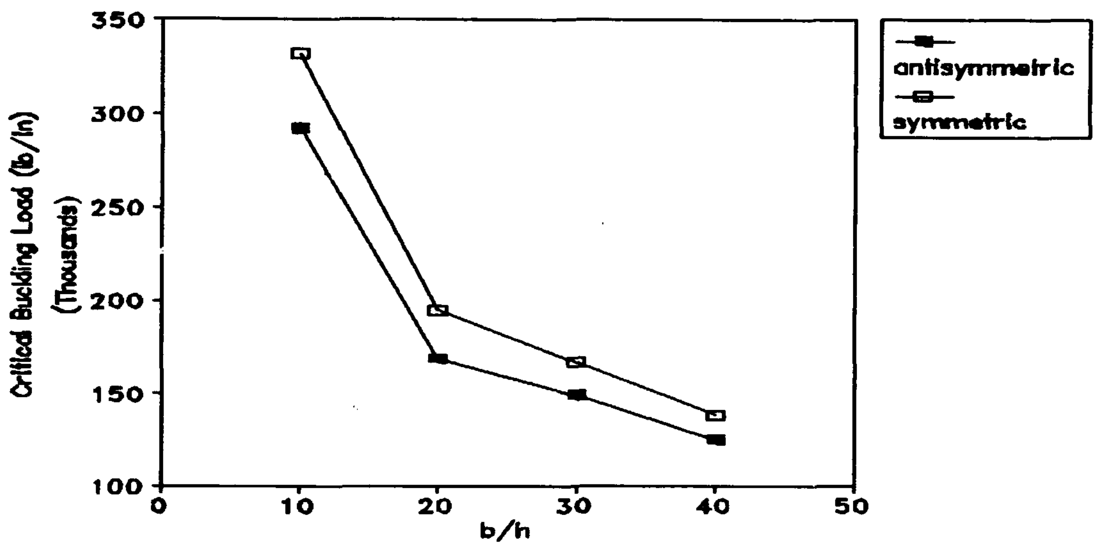

Figure 40. Comparison of Critical Buckling Loads for $[0/90]_s$ vs $[0/90]_{as}$ Clamped Boundary, h/R=1/20

The buckling loads are on the order of 10% lower for the antisymmetric case for all panel dimensions investigated. These results are as expected, and would appear to indicate good results for this laminate.

The natural frequencies do not exhibit similar behavior, however. The frequencies show decreases for nearly all cases. However, for b/h = 10 and 40 in Figure 41, and for b/h = 10 in Figure 42, the frequencies are seen to be larger for the nonsymmetric case, although they are very close for all cases, differing by less than 5%. The same was true for the simply supported antisymmetric laminates.

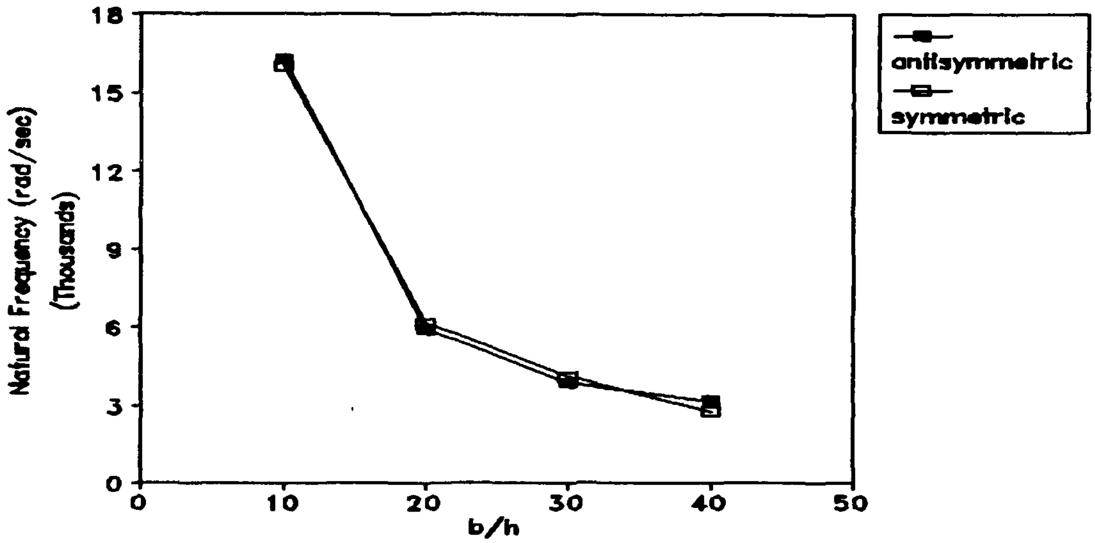

Figure 41. Comparison of Natural Frequencies for $[0/90]_s$ vs $[0/90]a_s$ Clamped Boundary, h/R=1/20

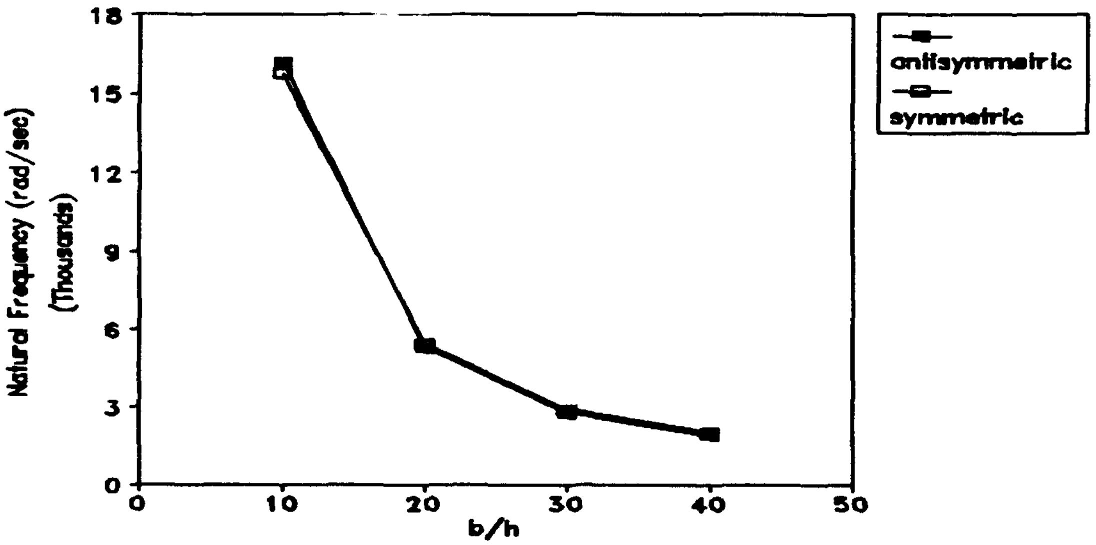

Figure 42. Comparison of Natural Frequencies for $[0/90]_s$ vs $[0/90]_{as}$ Clamped Boundary, h/R=1/50

Figures 43 and 44 show some typical results for the $[\pm 45]_{as}$ clamped laminate compared against the symmetric ply layup. Again, the buckling loads and frequencies are all higher for the antisymmetric case. The frequencies are again very close for both laminates.

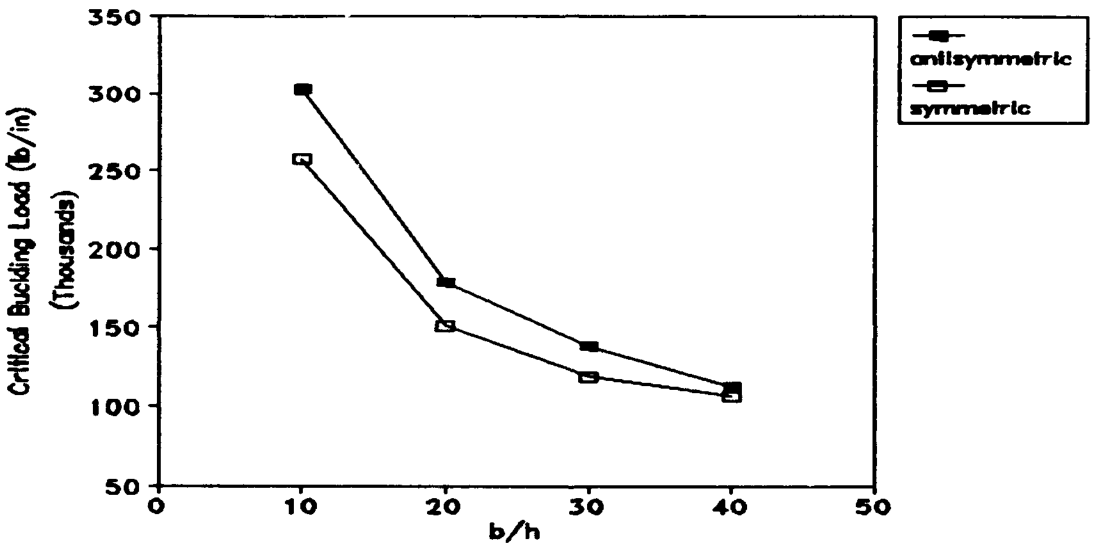

Figure 43. Comparison of Critical Buckling Loads for $[\pm 45]_s$ vs $[\pm 45]_{as}$ Clamped Boundary, h/R=1/20

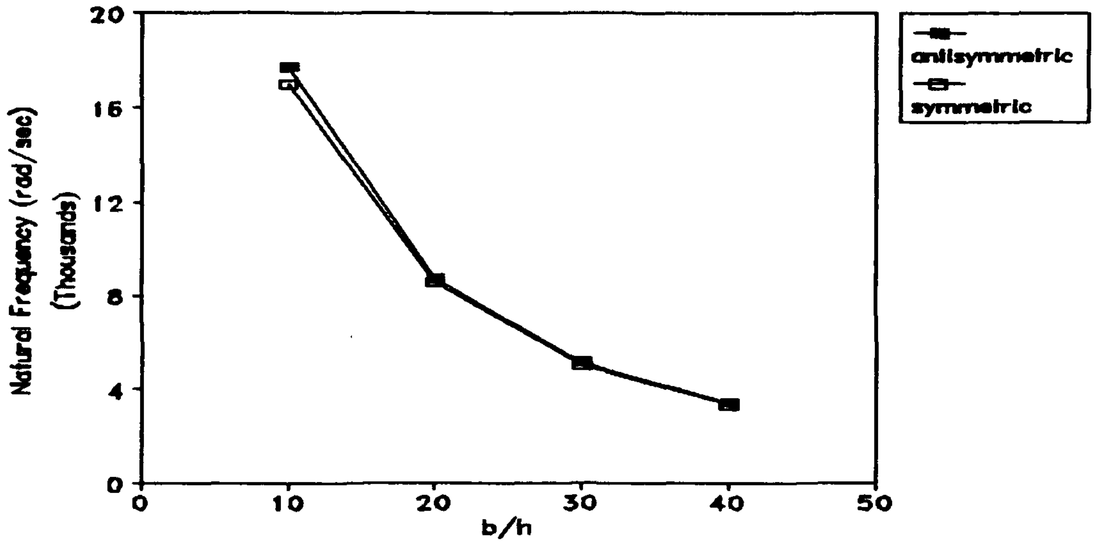

Figure 44. Comparison of Natural Frequencies for $[\pm45]_s$ vs $[\pm45]_{as}$ Clamped Boundary, h/R=1/20

Clamped-Simple Boundary Condition

This boundary condition assumes circumferential edges of the panel clamped, and the longitudinal edges simply supported. Bowlus and Reams investigated the effects of this type of boundary on flat plates assuming Mindlin type shear relations (2; 18). No results for a cylindrical shell could be found using the same higher order shear theory applied in this work.

Only two ply layups were investigated for this boundary condition, the $[0/90]_s$ and the $[\pm 45]_s$. Figures 45 through 50 show typical results for each of the layups. They show good convergence characteristics and follow the same trends established by the other boundary conditions, where the frequencies are seen to converge slightly faster than the buckling loads.

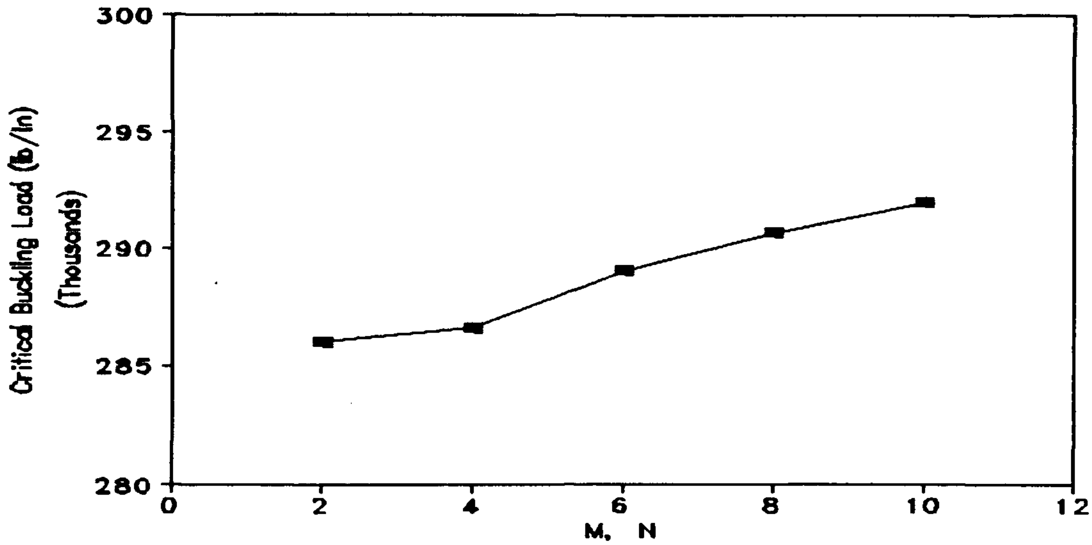

Figure 45. Convergence Characteristics for Critical Buckling Loads, Clamped-Simple Boundary, $[0/90]_s$, a=b=10, h/R=1/20

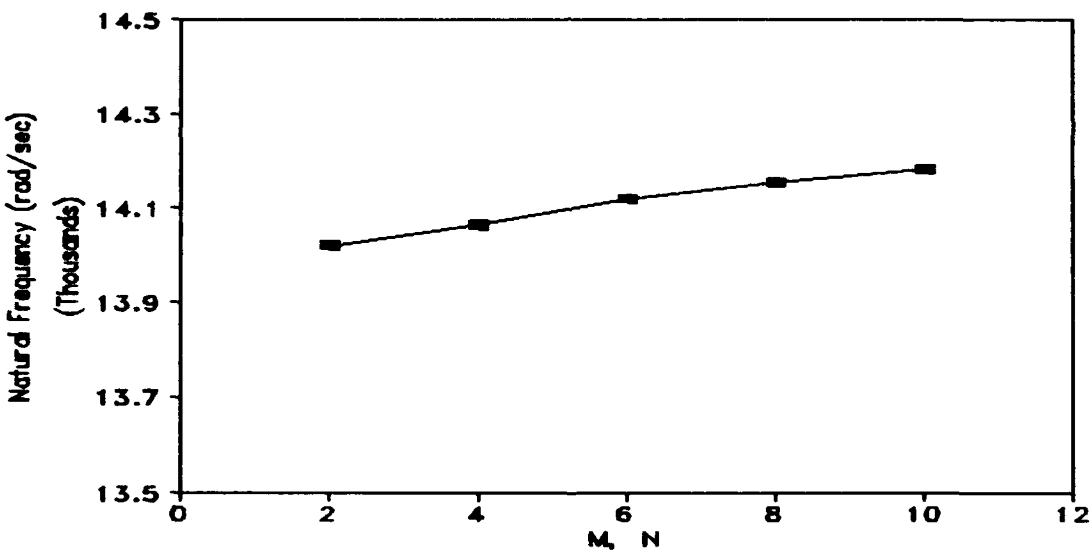

Figure 46. Convergence Characteristics for Natural Frequencies, Clamped-Simple Boundary, $[0/90]_s$, a=b=10, h/R=1/20

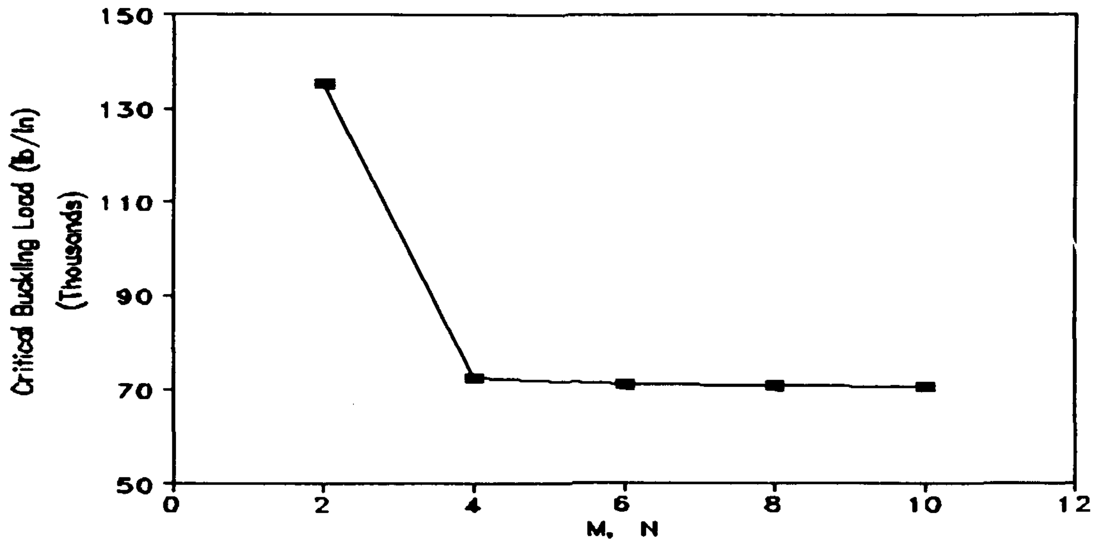

Figure 47. Convergence Characteristics for Critical Buckling Loads, Clamped-Simple Boundary, $[0/90]_s$, a=b=40, h/R=1/50

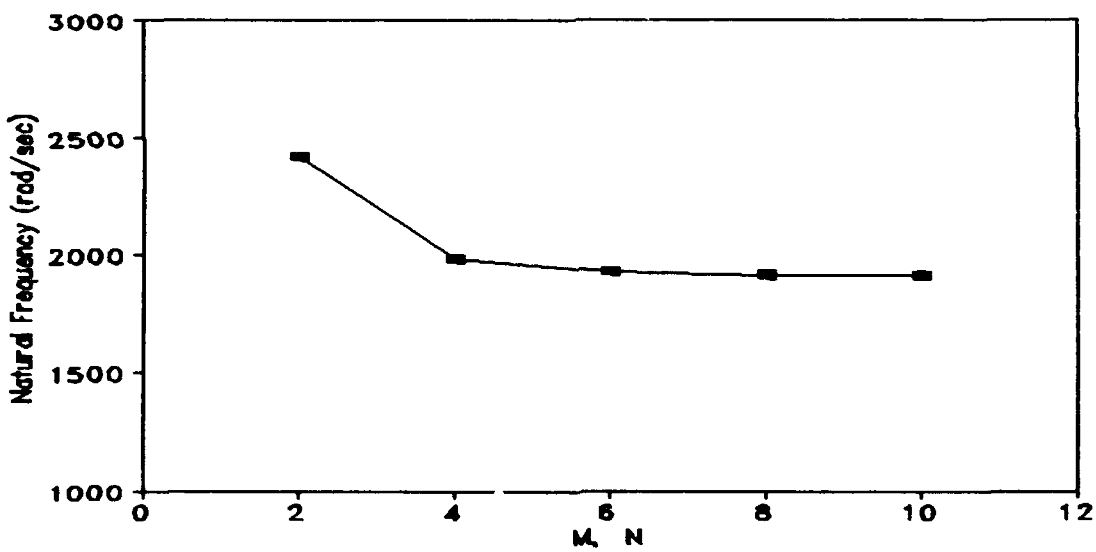

Figure 48. Convergence Characteristics for Natural Frequencies, Clamped-Simple Boundary, $[0/90]_s$, a=b=40, h/R=1/50

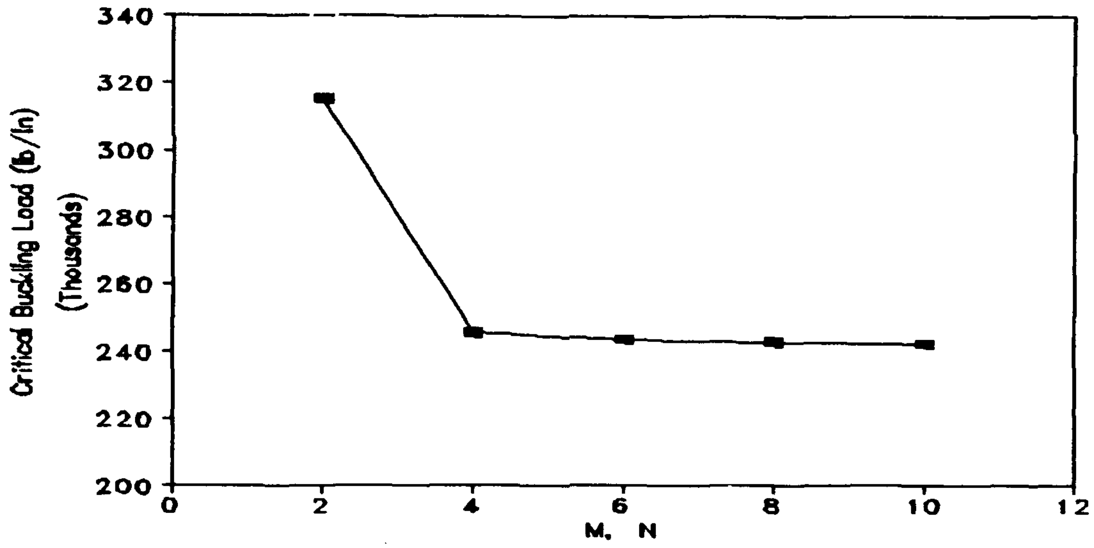

Figure 49. Convergence Characteristics for Critical Buckling Loads, Clamped-Simple Boundary, $[\pm 45]_s$, a=b=10, h/R=1/50

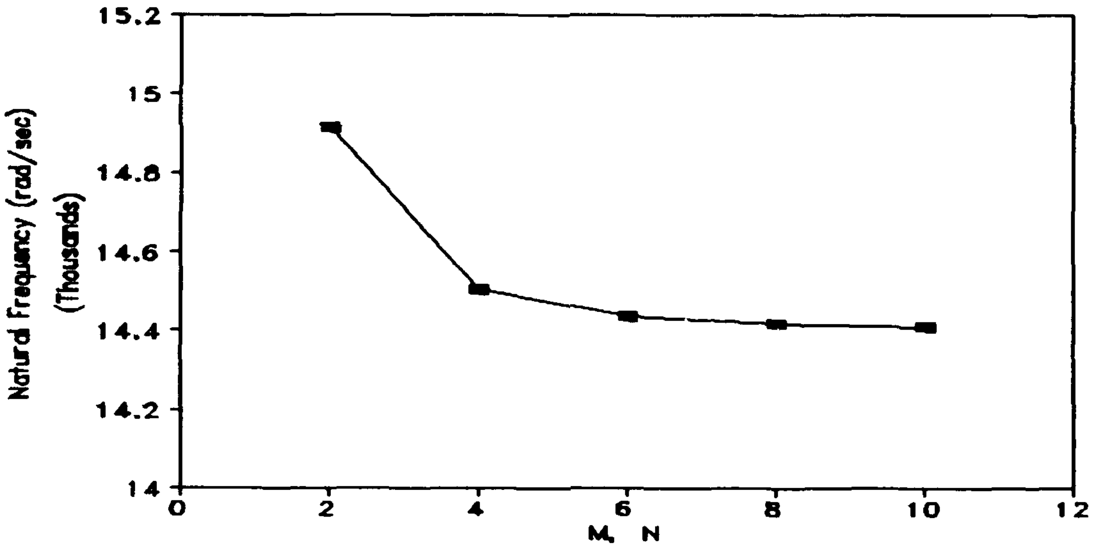

Figure 50. Convergence Characteristics for Natural Frequencies, Clamped-Simple Boundary, $[\pm 45]_s$, a=b=10, h/R=1/50

Figure 51 through 54 present the effects of curvature on the buckling loads and frequencies for both laminates. Just as for all previous cases, the loads and frequencies decrease with increased span to thickness ratios, with deeper shells less affected than flat plates. The effect is much greater for the buckling loads.

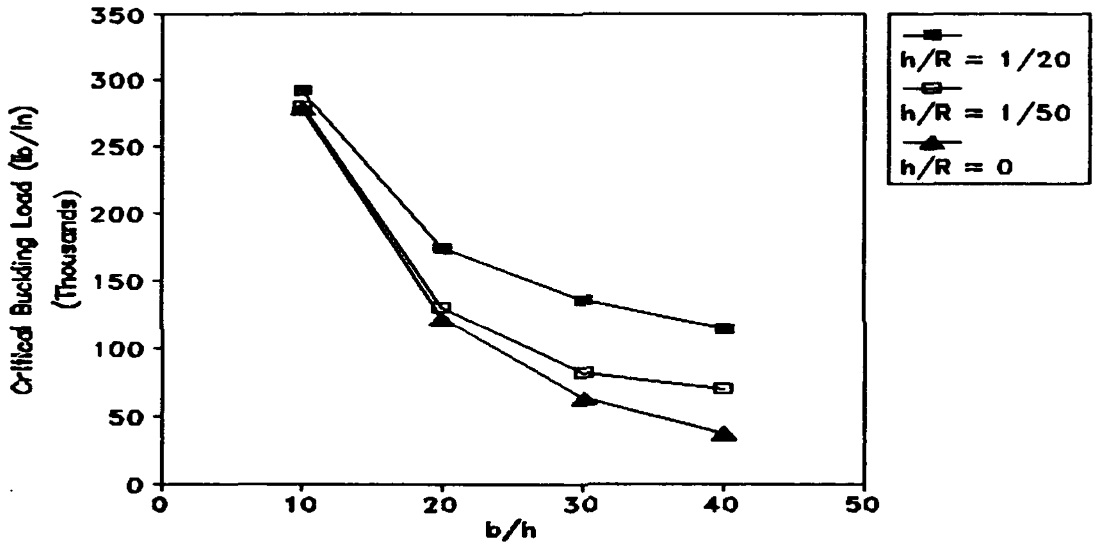

Figure 51. Curvature Effects on Critical Buckling Load, Clamped-Simple Boundary, $[0/90]_s$

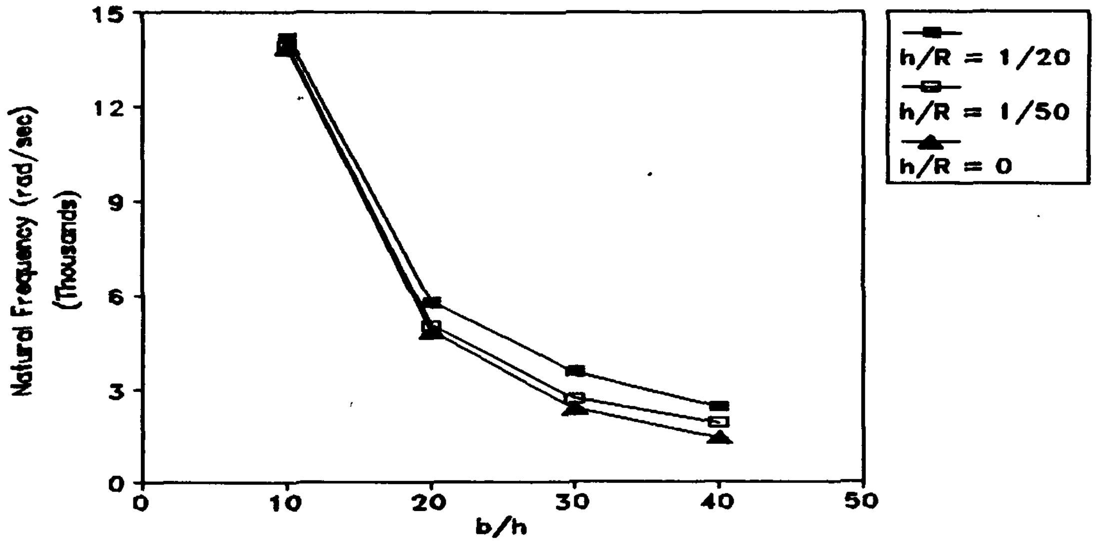

Figure 52. Curvature Effects on Natural Frequency, Clamped-Simple Boundary, $[0/90]_s$

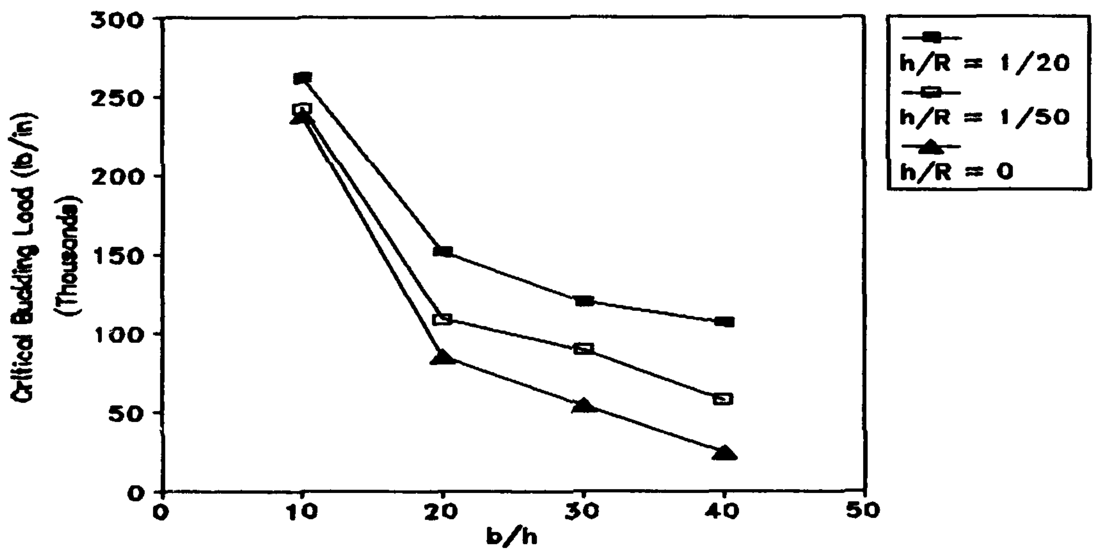

Figure 53. Curvature Effects on Critical Buckling Load, Clamped-Simple Boundary, $[\pm 45]_s$

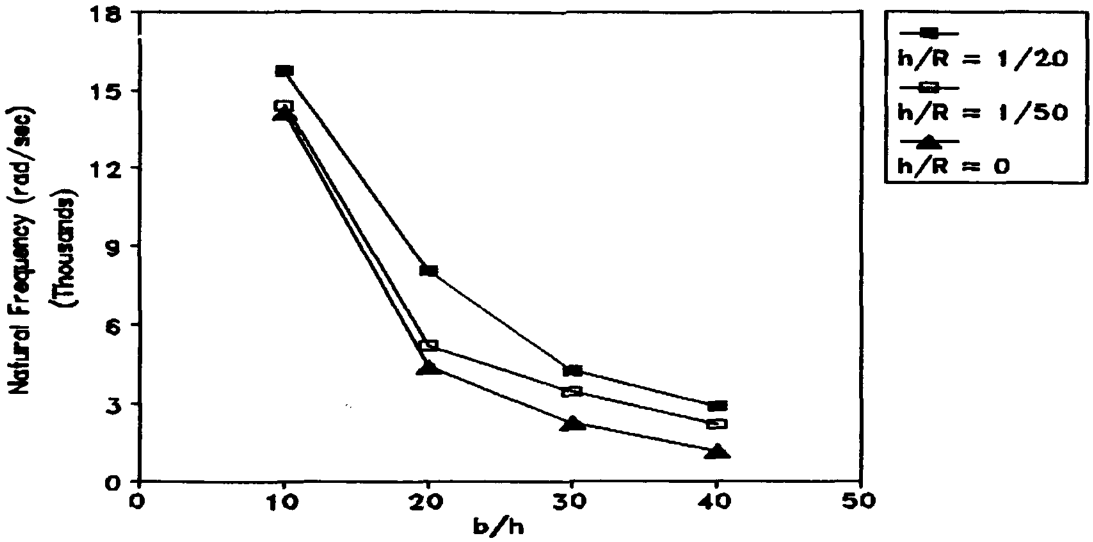

Figure 54. Curvature Effects on Natural Frequency, Clamped-Simple Boundary, $[\pm 45]_s$

Bowlus and Reams presented their results in nondimensionalized form, for comparisons to the other boundary conditions. They both showed that the buckling loads and frequencies for the clamped-simple boundary conditions fell between those for the simple and clamped boundary conditions. The same results are found for this analysis for the cylindrical shell.

Nondimensionalized results from both Reams and Bowlus indicate the magnitudes of the critical buckling loads and natural frequencies for this boundary condition fall between those for simply supported on all sides and clamped on all sides. The same results are found for the shells examined here, with one exception.

Figures 55 through 61 compare the critical buckling loads and natural frequencies for the $[0/90]_s$ laminate for different curvatures. Despite the anomalous behavior of the simply supported points described earlier, this indicates that for this ply layup, the buckling loads for the clamped-simple boundary seem to more closely emulate the purely clamped boundary. The results fall in between the simple and clamped cases, but there is quite a difference between the simple boundary condition and the clamped-simple case, especially for lower b/h ratios. This difference is also indicated in the natural frequencies, though the effect is not as obvious. This behavior is due to the clamping along the circumferential direction dominating the behavior of the laminate. The results for the [±45] ply layup do not show quite the same behavior. Examining the case for a radius of 20, see Figure 59, it is seen that the buckling loads for the clamped-simple boundary are slightly higher than both the clamped and simple boundaries. At the larger radius, see Figure 61, the loads fall below those for the clamped case. However, the differences between the results for each boundary condition are very small, on the on the order of a few percentages, that this behavior could be neglected for practical purposes. The results for the natural frequencies of the $[\pm 45]_s$ laminates are also very close for all boundary conditions. Figure 60

shows the clamped-simple results falling between the two, with no obvious inclinations to one side.

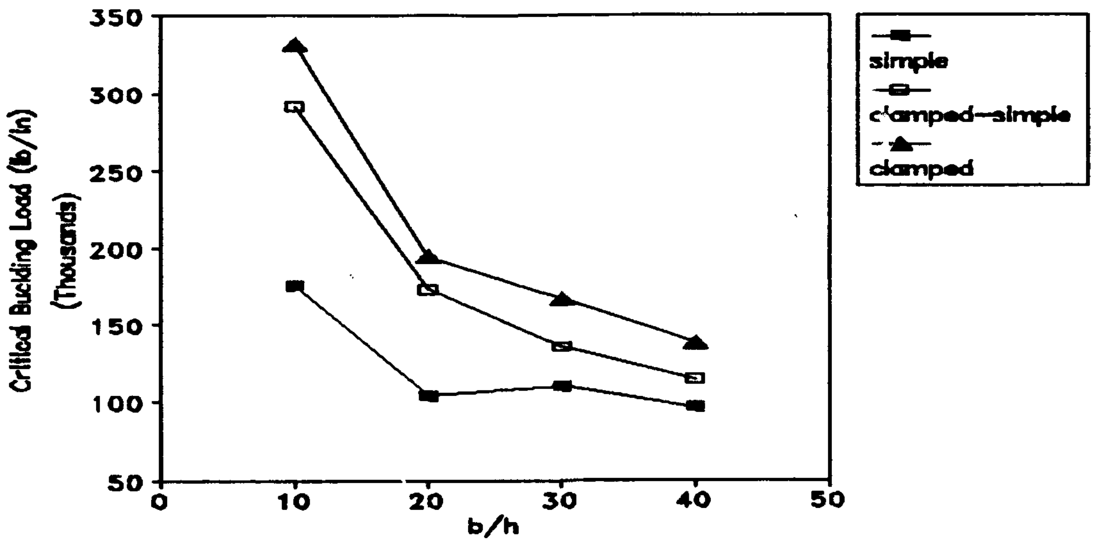

Figure 55. Comparison of Critical Buckling Loads for Different Boundary Conditions, $[9/90]_s$, h/R = 1/20

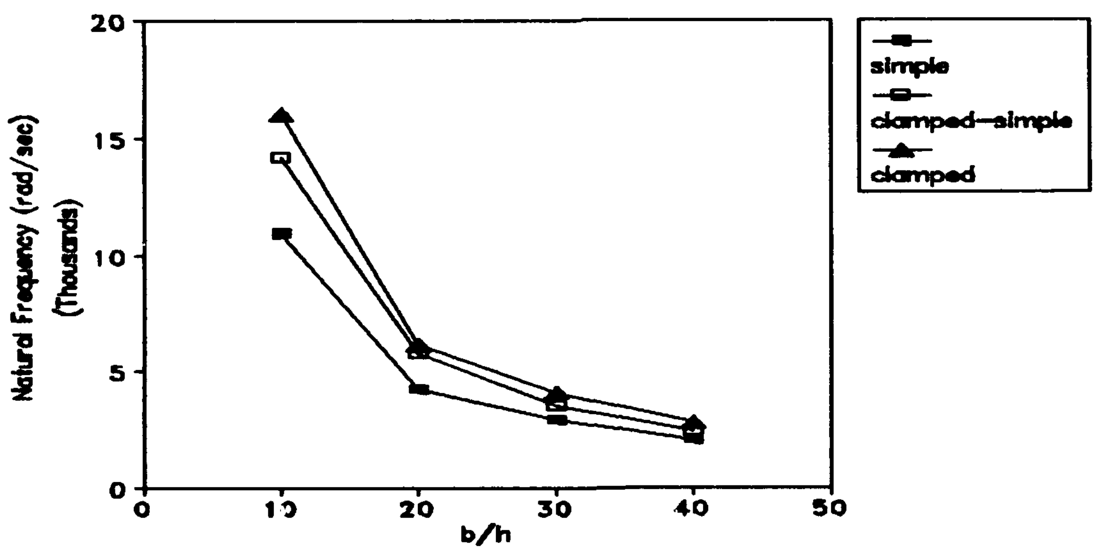

Figure 56. Comparison of Natural Frequencies for Different Boundary Conditions, $[0/90]_s$, h/R = 1/20

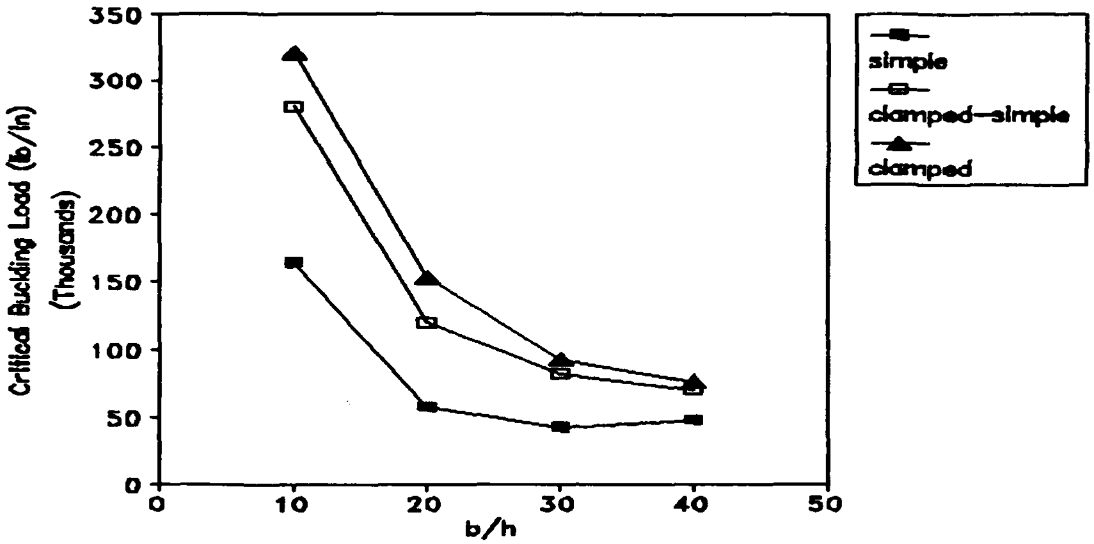

Figure 57. Comparison of Critical Buckling Load for Different Boundary Conditions, $[0/90]_s$, $h/R = 1/50$

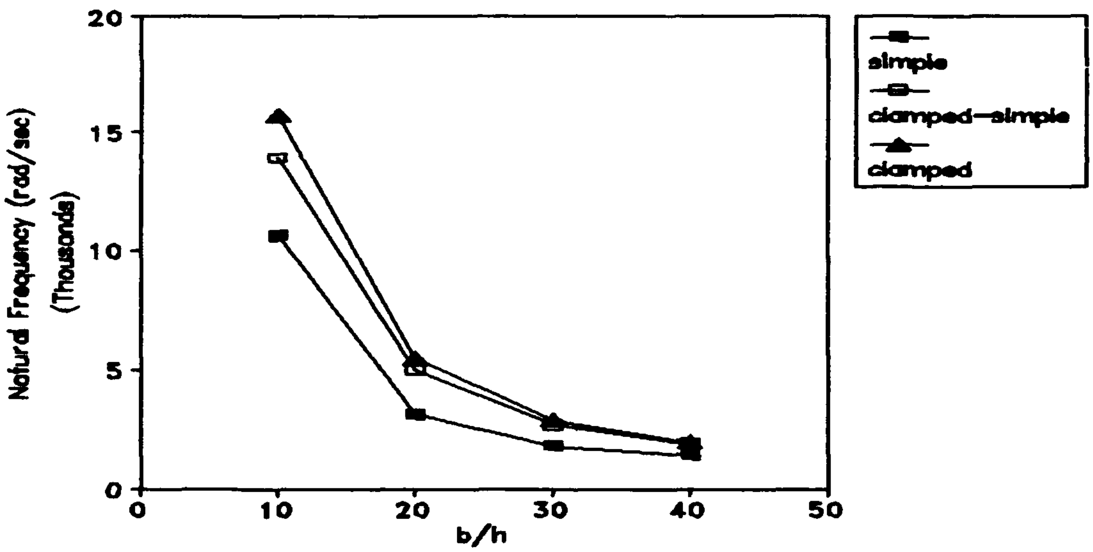

Figure 58. Comparison of Natural Frequencies for Different Boundary Conditions, $[0/90]_s$, $h/R = 1/50$

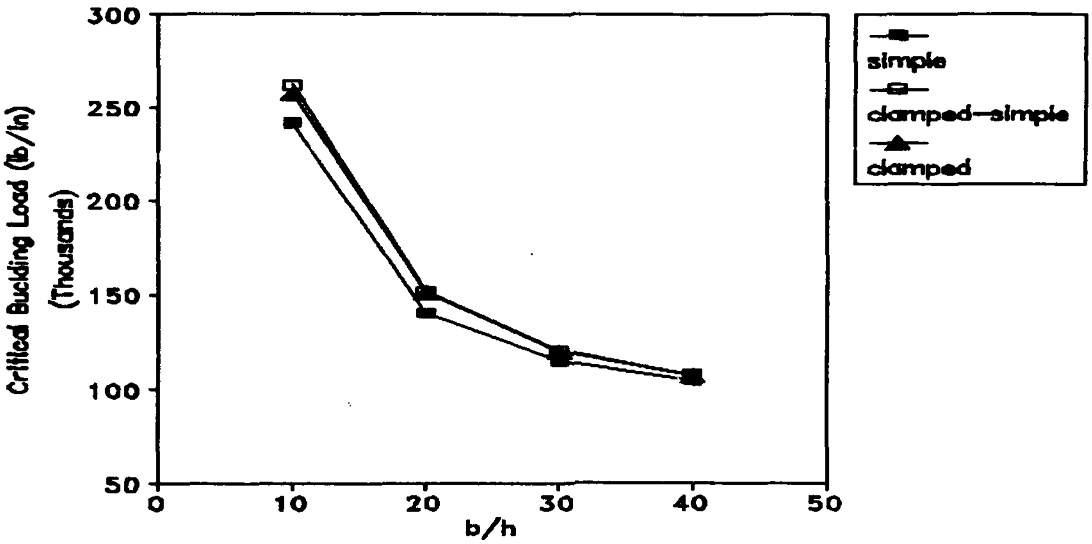

Figure 59. Comparison of Critical Buckling Loads for Different Boundary Conditions, $[\pm 45]_s$, h/R = 1/20

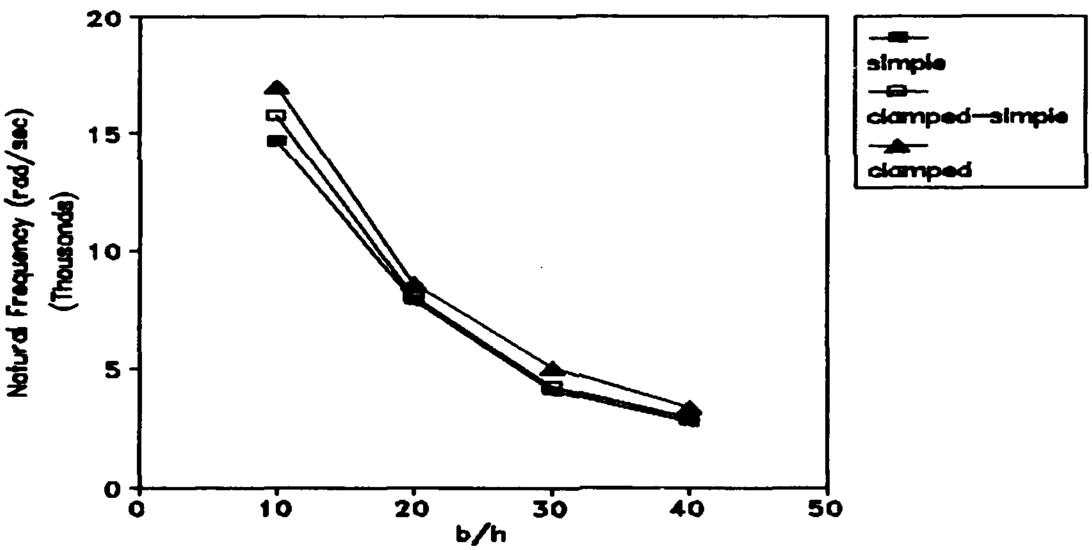

Figure 60. Comparison of Natural Frequencies for Different Boundary Conditions, $[\pm 45]_s$, h/R = 1/20

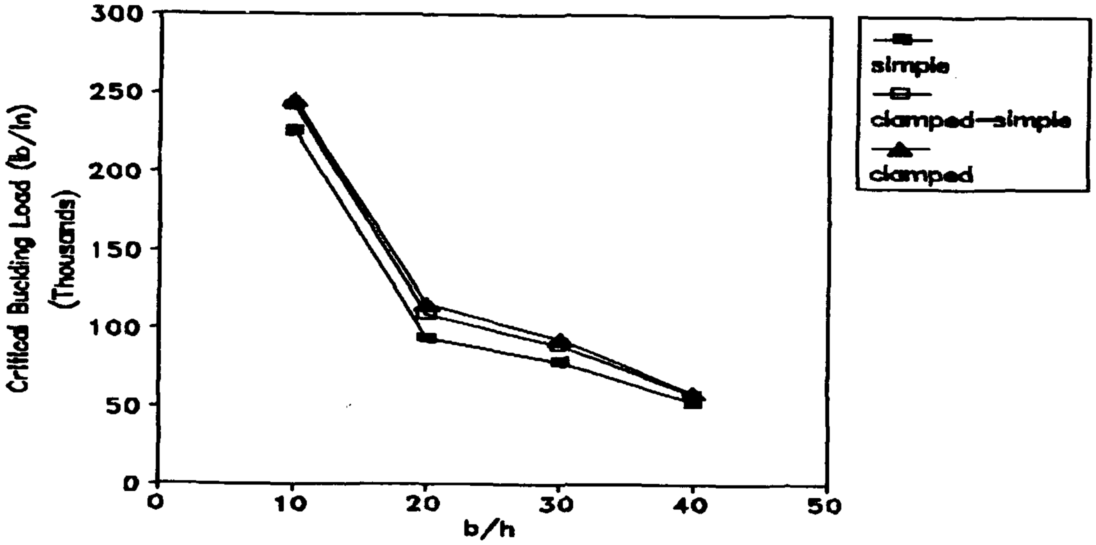

Figure 61. Comparison of Critical Buckling Loads for Different Boundary Conditions, $[\pm 45]_s$, $h/R = 1/50$

IV. *Summary and Conclusions*

A theory has been developed for general nonsymmetric laminated shells, including a parabolic transverse shear distribution. A computer program was developed to solve for the critical buckling loads and natural frequencies.

A variety of symmetric and nonsymmetric laminates were studied, with simply supported, clamped, and clamped-simple boundary conditions.

The following conclusions were reached from the results of this analysis.

First, the solutions generated from the Galerkin technique had excellent convergence characteristics for all laminates investigated. Convergence was slower for the buckling problems and for the clamped and clamped-simple boundary conditions. There were some indications for $M = N = 10$ terms of increases in the buckling loads and frequencies. These increases were very small, and divergence seems unlikely. However, increasing the maximum number of terms in the solutions is recommended just to make sure there is in reality no divergence. There were some results for the simply supported boundary that did not follow the trends established. These points are assumed to be incorrect, as the majority of the results follow consistent, logical trends.

The curvature of the shells was varied to measure the effects for each laminate, symmetric and nonsymmetric, for each of the boundary conditions. The results showed that the deeper the shell, the higher the stiffness. Differences between buckling loads for deep shells and flat plates was significant. The frequencies were also affected, but not to the extent of the buckling problem.

The natural frequencies obtained for each boundary condition were generally greater for the $[\pm45]_s$ laminates than for the $[0/90]_s$, due to the presence of additional stiffness terms. This is not the case for the buckling problems. The $[0/90]_s$ laminates yielded larger buckling loads for the clamped and clamped-simple boundary condition, and also for small a/h ratios in the simply supported boundary.

The $[0/\pm45/90]_s$ laminates produced buckling loads and frequencies consistently higher than the $[0/90]_s$ laminates for both the simple and clamped boundaries. The buckling loads were higher than the loads for the $[\pm45]_s$ laminates for both boundaries considered, while the frequencies were lower for the simply supported case, and some of the clamped results.

The antisymmetric results are not as expected. Only for one case did the antisymmetric buckling loads fall consistently below the symmetric case, the clamped [0/90] laminate. The majority of the laminates investigated, including simple and clamped boundaries with $[0/90]_{as}$ and $[\pm45]_{as}$ ply layups,

resulted in buckling loads and frequencies that were higher for the antisymmetric laminate.

Finally, the clamped-simple boundary condition produced results that agreed with previous work. The results for the critical buckling loads and natural frequencies for the most part fell between the values obtained from a purely simply supported and a purely clamped boundary. Clamping the circumferential length resulted in the buckling loads greatly influenced by the clamped behavior, seeming to follow the behavior of a purely clamped boundary. The frequencies did not seem to be as affected.

Much work remains to be done. The behavior of the antisymmetric laminates versus the symmetric found here may be explained by the effects of the large number of plies as discussed in the previous chapter. Further investigations should be done to determine if this is the case.

Another logical continuation from this work would be to include throughout the development the through the thickness shear strain ε_z, to determine its effects.

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Appendix A. Transverse Shear

This appendix shows the development of the displacement functions based on the boundary condition of zero transverse shear strain at the top and bottom surface of the laminate.

This appendix focuses on γ_{yz}. The development relating to γ_{xz} is the same.

The strain expression of Eq (2), assuming z/R ≈ 0 is shown below.

$$\gamma_{yz} = v_{,z} + w_{,y} - \frac{v}{R} \tag{A.1}$$

Substituting the displacement relations of Eq (1) int Eq (A.1) gives the following:

$$\gamma_{yz} = \frac{v_o}{R} + \psi_y + 2z\phi_2 + 3z^2\theta_2 + w_{o.y} + z\xi_{,y} - \left[1 + \frac{z}{R}\right]\frac{v_o}{R} - \frac{z}{R}\psi_y - \frac{z^2}{R}\phi_2 - \frac{z^3}{R}\theta_2 \tag{A.2}$$

The above simplifies with z/R ≈ 0 to

$$\gamma_{yz} = \psi_y + 2z\phi_2 + 3z^2\theta_2 + w_{o,y} + z\xi_{,y} \tag{A.3}$$

To satisfy the boundary conditions, evaluate Eq (A.3) at z = ±h/2 and set equal to zero:

$$\psi_y + h\phi_2 + \frac{3h^2}{4}\theta_2 + w,_{o,y} + \frac{h}{2}\xi = 0$$
$$\psi_y - h\phi_2 + \frac{3h^2}{4}\theta_2 + w,_{o,y} - \frac{h}{2}\xi = 0 \tag{A.4}$$

Adding the above expressions results in the following:

$$\theta_2 = -\frac{4}{3h^2}(\psi_y + w_{o,y})$$
$$\phi_2 = -\frac{1}{2}\xi_y \tag{A.5}$$

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Appendix B. Computer Code

This appendix contains a listing of the FORTRAN code used for this thesis. The program included is that for the simply supported boundary condition. The codes for the clamped and clamped-simple boundary conditions are identical, except for the Galerkin equations used in the GALERK subroutine. These additional subroutines are not included here, as the equations are given in the Theoretical Development.

```
      Program MAINTHESIS
c
c CAPT KATHLEEN V. TIGHE
c
c GAE-91D
c
c THE DETERMINATION OF THE FUNDAMENTAL NATURAL FREQUENCY AND
c CRITICAL BUCKLING LOAD OF AN ANISOTROPIC LAMINATED CIRCULAR
c CYLINDRICAL SHELL PANEL INCLUDING THE EFFECTS OF PARABOLIC
c TRANSVERSE SHEAR DEFORMATION AND ROTARY INERTIA
C
c THESIS ADVISOR: DR ANTHONY PALAZOTTO
c
c-----------------------------------------------------------------
c
c   INITIALIZATION
c
      Double Precision A, B, R, H, PI, A11, A12, A22, A16, A26, A66,
     &  A44, A45, A55, D11, D12, D22, D16, D26, D66, D44, D45, D55,
     &  F11, F12, F22, F16, F26, F66, F44, F45, F55, H11, H12, H22,
     &  H16, H26, H66, J11, J12, J22, J16, J26, J66, B11, B12, B22,
     &  B16, B26, B66, E11, E12, E22, E16, E26, E66, G11, G12, G22,
     &  G16, G26, G66, I11, I12, I22, I16, I26, I66, TPLY,
     &  THETA(100), E1, E2, GM12, V12, V21, G13, G23, STIFF(500,500),
     &  MASS(500,500), BETA(500), RHO, REVEC(100)
c
      Double Complex ALPHA(500), EVAL(500), EVEC(500,500)
c
c   WORKSPACE ALLOCATION FOR IMSL
c
      Common /WORKSP/ RWKSP
      Real RWKSP(3003026)
      Call IWKIN(3003026)
c
      Open (Unit=1,File='main.in',Status='OLD')
      Open (Unit=2,File='main.out',Status='NEW')
c
c   IS THIS A VIBRATION PROBLEM OR A BUCKLING PROBLEM ?
C   NBUCVIB = 1 ; VIBRATION.     NBUCVIB = 2; BUCKLING.
C
      Read (1,1300) NBUCVIB
```

```
c
c------------------------------------------------------------
c
c   READ SHELL PANEL DIMENSIONS AND LAMINATE DATA
c
c   DIMENSIONAL DATA
c
C   LENGTH IN THE X DIRECTION (LONGITUDINAL AXIS)
    Read (1,1000) A
c   LENGTH IN THE Y DIRECTION (CIRCUMFERENCIAL AXIS)
    Read (1,1000) B
c   RADIUS OF CURVATURE
    Read (1,1000) R
c   LAMINATE THICKNESS
    Read (1,1000) H
    PI = 3.1415926553588793
c   LENGTH TO SPAN RATIO AND THICKNESS RATIO
    AOVERB = A / B
    HOVERR = H / R
    AOVERH = A / H
    BOVERH = B / H
c
c LAMINATE DATA
c
c NUMBER OF PLYS IN THE LAMINATE
    Read (1,1300) NPLYS
c THICKNESS OF EACH PLY IN THE LAMINATE
    TPLY = H / NPLYS
c ORIENTATION ANGLE OF EACH PLY IN THE LAMINATE
    Do 10 I = 1, NPLYS
     Read (1,1000) THETA(I)
  10 Continue
c  MATERIAL PROPERTIES OF EACH PLY
    Read (1,1000) E1
    Read (1,1000) E2
    Read (1,1000) GM12
    Read (1,1000) V12
    Read (1,1100) RHO
    V21 = V12 * E2 / E1
c FOR THIS THESIS, G13 AND G23 WILL HAVE THE FOLLOWING VALUES :
    G13 = GM12
    G23 = 0.8 * GM12
c
```

```
c-----------------------------------------------------------
c
c  WRITE SHELL PANEL DIMENSIONS AND LAMINATE DATA
c
      Write (2,1400)
      If (NBUCVIB.EQ.1) Then
       Write (2,1500)
      Else
       Write (2,1600)
      End If
      Write (2,1700)
      Write (2,1800)
      Write (2,1900) A, B, AOVERB
      Write (2,2000) H, AOVERH, BOVERH
      Write (2,2100) R, HOVERR
      Write (2,2200)
      Write (2,2300)
      Do 20 I = 1, NPLYS
       Write (2,2400) THETA(I)
   20 Continue
      Write (2,2500) NPLYS, H
      Write (2,2600) TPLY
      Write (2,2700) E1, E2
      Write (2,2800) GM12
      Write (2,2900) G13, G23
      Write (2,3000) V12, V21
      Write (2,3100) RHO
c
c-----------------------------------------------------------
c
c CALCULATE THE BENDING, EXTENSIONAL, AND HIGHER ORDER
c STIFFNESS ELEMENTS FOR A GENERAL NON-SYMMETRIC LAMINATE.
C
      Call LAMINAT(NPLYS,TPLY,THETA,E1,E2,GM12,V12,V21,G13,G23,PI,H,A11,
     &  A12,A22,A16,A26,A66,A44,A45,A55,D11,D12,D22,D16,D26,D66,D44,
     &  D45,D55,F11,F12,F22,F16,F26,F66,F44,F45,F55,H11,H12,H22,H16,
     &  H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,B22,B16,B26,B66,E11,
     &  E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,I11,I12,I22,I16,
     &  I26,I66)
c
c-----------------------------------------------------------
c
c  WRITE LAMINATE STIFFNESS ELEMENTS
```

```
c
    Write (2,3200)
    Write (2,3300) A11, A12, A22
    Write (2,3400) A16, A26, A66
    Write (2,3500) A44, A45, A55
    Write (2,3600)
    Write (2,3700) B11, B12, B22
    Write (2,3800) B16, B26, B66
    Write (2,3900)
    Write (2,4000) D11, D12, D22
    Write (2,4100) D16, D26, D66
    Write (2,4200) D44, D45, D55
    Write (2,4300)
    Write (2,4400)
    Write (2,4500) E11, E12, E22
    Write (2,4600) E16, E26, E66
    Write (2,4700) F11, F12, F22
    Write (2,4800) F16, F26, F66
    Write (2,4900) F44, F45, F55
    Write (2,5000) G11, G12, G22
    Write (2,5100) G16, G26, G66
    Write (2,5200) H11, H12, H22
    Write (2,5300) H16, H26, H66
    Write (2,5400) I11, I12, I22
    Write (2,5500) I16, I26, I66
    Write (2,5600) J11, J12, J22
    Write (2,5700) J16, J26, J66
c
-----------------------------------------------------------------
c
c   DETERMINE THE DIMENSION OF THE MASS AND STIFFNESS MATRICES.
c
    Read (1,1300) MMAX
    MSIZE = 5 * MMAX * MMAX
    MSIZESQ = MMAX * MMAX
    If (NBUCVIB.EQ.1) Then
      Write (2,5800)
      Write (2,6000) MMAX, MSIZE, MSIZE
      Write (2,6100)
    Else
      Write (2,5900)
      Write (2,6200)
      Write (2,6300)
```

```
    Write (2,6000) MMAX, MSIZE, MSIZE
   End If
c-----------------------------------------------
c
c USING THE BENDING, EXTENSIONAL, AND HIGHER ORDER STIFFNESS
c ELEMENTS AND THE SHELL PANEL PHYSICAL CHARACTERISTICS AS
c INPUTS, CALCULATE THE STIFFNESS AND MASS MATRICES AND THEN
c FIND THE NATURAL FREQUENCIES AND/OR AXIAL BUCKLING LOAD AND
c THEIR RESPECTIVE MODE SHAPES.
C
   Call GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,D11,
  &  D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,
  &  F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,
  &  B12,B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,
  &  G26,G66,I11,I12,I22,I16,I26,I66,NBUCVIB,MMAX,MSIZE,RHO,STIFF,
  &  MASS,BETA,ALPHA,EVAL,EVEC,MSIZESQ,REVEC)
c-----------------------------------------------
   Stop
c
c FORMAT STATEMENTS
c
 1000 Format (F15.5)
 1100 Format (D22.15)
 1200 Format (E12.5)
 1300 Format (I5)
 1400 Format (////,5X,
  &  'ANISOTROPIC LAMINATED CIRCULAR CYLINDRICAL SHELL PANEL')
 1500 Format (//,5X,'VIBRATION PROBLEM')
 1600 Format (//,5X,'BUCKLING PROBLEM')
 1700 Format (//,5X,'S2 SIMPLY SUPPORTED BOUNDARY CONDITIONS')
 1800 Format (///,5X,'SHELL PANEL DIMENSIONS (in.)')
 1900 Format (/,5X,'a = ',1X,F6.2,4X,'b = ',1X,F6.2,4X,'a/b = ',1X,F6.2)
 2000 Format (/,5X,'h = ',1X,F4.2,4X,'a/h = ',1X,F6.2,4X,'b/h = ',1X,F6
  &  .2)
 2100 Format (/,5X,'R = ',1X,E12.5,6X,'h/R = ',1X,F10.8)
 2200 Format (//,5X,'SHELL PANEL LAMINATE DATA')
 2300 Format (//,5X,'LAMINATE PLY LAYUP (DEGREES)')
 2400 Format (/,30X,F7.2)
 2500 Format (/,3X,I3,2X,'PLYS IN THIS',2X,F4.2,2X,'in. THICK LAMINATE')
 2600 Format (/,5X,'EACH PLY IS',1X,E12.5,2X,'ins. THICK')
 2700 Format (/,5X,'ELASTICITY MODULII (psi) : E1 = ',E12.5,2X,'E2= '
  &  ,E12.5)
 2800 Format (/,5X,'IN PLANE SHEAR MODULUS (psi):  G12 = ',E12.5)
```

```
2900 Format (/,5X,'TRANSVERSE SHEAR MODULII (psi):  G13 = ',E12.5,2X,
  &   'G23 = ',E12.5)
3000 Format (/,5X,'POISSONS RATIOS: V12 = ',1X,F6.4,3X,'V21 = ',1X,F6.4
  &  )
3100 Format (/,5X,'MASS DENSITY (LB*SEC^2/IN^4):  RHO =  ',1X,D18.11)
3200 Format (///,5X,'EXTENSIONAL STIFFNESS ELEMENTS (lb/in)')
3300 Format (/,5X,'A11 = ',F15.3,3X,'A12 = ',F15.3,3X,'A22 = ',F15.3)
3400 Format (/,5X,'A16 = ',F15.3,3X,'A26 = ',F15.3,3X,'A66 = ',F15.3)
3500 Format (/,5X,'A44 = ',F15.3,3X,'A45 = ',F15.3,3X,'A55 = ',F15.3)
3600 Format (///,5X,'COUPLING STIFFNESS ELEMENTS (lb?)')
3700 Format (/,5X,'B11 = ',F15.3,3X,'B12 = ',F15.3,3X,'B22 = ',F15.3)
3800 Format (/,5X,'B16 = ',F15.3,3X,'B26 = ',F15.3,3X,'B66 = ',F15.3)
3900 Format (//,5X,'BENDING STIFFNESS ELEMENTS (lb * in)')
4000 Format (/,5X,'D11 = ',F15.3,3X,'D12 = ',F15.3,3X,'D22 = ',F15.3)
4100 Format (/,5X,'D16 = ',F15.3,3X,'D26 = ',F15.3,3X,'D66 = ',F15.3)
4200 Format (/,5X,'D44 = ',F15.3,3X,'D45 = ',F15.3,3X,'D55 = ',F15.3)
4300 Format (//,5X,'HIGHER ORDER STIFFNESS ELEMENTS')
4400 Format (5X,'Fij (in * lb^3), Hij(in * lb^5 ), Jij(in * lb^7)')
4500 Format (/,5X,'E11 = ',F15.3,3X,'E12 = ',F15.3,3X,'E22 = ',F15.3)
4600 Format (/,5X,'E16 = ',F15.3,3X,'E26 = ',F15.3,3X,'E66 = ',F15.3)
4700 Format (//,5X,'F11 = ',F15.3,3X,'F12 = ',F15.3,3X,'F22= ',F15.3)
4800 Format (/,5X,'F16 = ',F15.3,3X,'F26 = ',F15.3,3X,'F66 = ',F15.3)
4900 Format (/,5X,'F44 = ',F15.3,3X,'F45 = ',F15.3,3X,'F55 = ',F15.3)
5000 Format (//,5X,'G11 = ',F15.3,3X,'G12 = ',F15.3,3X,'G22= ',F15.3)
5100 Format (/,5X,'G16 = ',F15.3,3X,'G26 = ',F15.3,3X,'G66 = ',F15.3)
5200 Format (//,5X,'H11 = ',F15.3,3X,'H12 = ',F15.3,3X,'H22= ',F15.3)
5300 Format (/,5X,'H16 = ',F15.3,3X,'H26 = ',F15.3,3X,'H66 = ',F15.3)
5400 Format (//,5X,'I11 = ',F15.3,3X,'I12 = ',F15.3,3X,'I22= ',F15.3)
5500 Format (/,5X,'I16 = ',F15.3,3X,'I26 = ',F15.3,3X,'I66 = ',F15.3)
5600 Format (//,5X,'J11 = ',F15.3,3X,'J12 = ',F15.3,3X,'J22= ',F15.3)
5700 Format (/,5X,'J16 = ',F15.3,3X,'J26 = ',F15.3,3X,'J66 = ',F15.3)
5800 Format (///,5X,
  &   'VIBRATION EIGENVALUE ANALYSIS - FIRST 10 MODES PRINTED')
5900 Format (///,5X,'BUCKLING EIGENVALUE ANALYSIS - ALL MODES PRINTED')
6000 Format (//,5X,'MMAX = NMAX = ',I2,5X,
  &   'STIFFNESS AND MASS/INERTIA MATRICES ARE ( ',I3,1X,' BY ',1X,
  &   I3,' ) ')
6100 Format (///,5X,'MODE NUMBER ',11X,' EIGENVALUE ',14X,
  &   'NATURAL FREQUENCY ( RAD/SEC )')
6200 Format (5X,'THE CRITICAL BUCKLING LOAD IS THE EIGENVALUE WITH')
6300 Format (5X,'THE SMALLEST ABSOLUTE VALUE ')
c-----------------------------------------------------------------
   End
```

```
c---------------------------------------------------------
   Subroutine LAMINAT(NPLYS,TPLY,THETA,E1,E2,GM12,V12,V21,G13,G23,PI,
   & H,A11,A12,A22,A16,A26,A66,A44,A45,A55,D11,D12,D22,D16,D26,D66,
   & D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,F45,F55,H11,H12,H22,
   & H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,B22,B16,B26,B66,
   & E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,I11,I12,I22,
   & I16,I26,I66)
c
c---------------------------------------------------------
c
c  THIS SUBROUTINE CALCULATES THE BENDING, COUPLINGC EXTENSIONAL,
C   AND HIGHER ORDER STIFFNESS ELEMENTS FOR THE LAMINATE.
c
c  THIS THESIS ASSUMES A HOMOGENEOUS LAMINATE -- MATERIAL
c  PROPERTIES ARE IDENTICAL FOR EACH PLY. THE THICKNESS IS THE SAME
c  FOR EACH PLY...ONLY THE ORIENTATION ANGLE CAN CHANGE.
c
c---------------------------------------------------------
   Double Precision H, PI, A11, A12, A22, A16, A26, A66, A44, A45,
   & A55, D11, D12, D22, D16, D26, D66, D44, D45, D55, F11, F12,
   & F22, F16, F26, F66, F44, F45, F55, H11, H12, H22, H16, H26,
   & H66, J11, J12, J22, J16, J26, J66, B11, B12, B22, B16, B26,
   & B66, E11, E12, E22, E16, E26, E66, G11, G12, G22, G16, G26,
   & G66, I11, I12, I22, I16, I26, I66, TPLY, THETA(100), E1, E2,
   & GM12, V12, V21, G13, G23, Q11, Q12, Q22, Q44, Q55, Q66,
   & QBAR11, QBAR12, QBAR16, QBAR22, QBAR26, QBAR44, QBAR45,
   & QBAR55, QBAR66, ZK, ZK1, TH(100), ZKO, ZK3, ZK5, ZK7, ZK9
c
c   REDUCED STIFFNESS ELEMENTS IN PRINCIPLE COORDINATES
   Q11 = E1 / (1.0-V12*V21)
   Q12 = V12 * E2 / (1.0-V12*V21)
   Q22 = E2 / (1.0-V12*V21)
   Q44 = G23
   Q55 = G13
   Q66 = GM12
c   INITIALIZE ALL STIFFNESS ELEMENTS TO ZERO
   A11 = 0.
   A12 = 0.
   A22 = 0.
   A16 = 0.
   A26 = 0.
   A66 = 0.
   A44 = 0.
```

A45 = 0.
A55 = 0.
B11 = 0.
B12 = 0.
B22 = 0.
B16 = 0.
B26 = 0.
B66 = 0.
D11 = 0.
D12 = 0.
D22 = 0.
D16 = 0.
D26 = 0.
D66 = 0.
D44 = 0.
D45 = 0.
D55 = 0.
E11 = 0.
E12 = 0.
E22 = 0.
E16 = 0.
E26 = 0.
E66 = 0.
F11 = 0.
F12 = 0.
F22 = 0.
F16 = 0.
F26 = 0.
F66 = 0.
F44 = 0.
F45 = 0.
F55 = 0.
G11 = 0.
G12 = 0.
G22 = 0.
G16 = 0.
G26 = 0.
G66 = 0.
H11 = 0.
H12 = 0.
H22 = 0.
H16 = 0.
H26 = 0.

```
      H66 = 0.
      I11 = 0.
      I12 = 0.
      I22 = 0.
      I16 = 0.
      I26 = 0.
      I66 = 0.
      J11 = 0.
      J12 = 0.
      J22 = 0.
      J16 = 0.
      J26 = 0.
      J66 = 0.
C-----------------------------------------------------------------
c     IN ORDER FROM THE FIRST PLY AT Z = - H/2 TO THE TOP PLY AT
c     z = + H/2, INPUT THE PLY ORIENTATION ANGLE, THETA. THEN IN
c     TURN CALCULATE THE QBARS AND THE STIFFNESS ELEMENTS FOR THAT
c     PLY. REPEAT THE PROCEDURE FOR ALL PLYS, THEN ADD THE PLY
c     STIFFNESS ELEMENTS TOGETHER TO GET THE LAMINATE STIFFNESS
c     ELEMENTS.
c     INITIALIZE Z TO THE BOTTOM OF THE LAMINATE
c-----------------------------------------------------------------
      ZK1 = -H / 2.0
      Do 10 I = 1, NPLYS
       TH(I) = THETA(I) * PI / 180.0
c-----------------------------------------------------------------
c     COMPUTE THE QBARS - TRANSFORMED REDUCED STIFFNESS ELEMENTS
c     IN  GLOBAL COORDINATES.
C-----------------------------------------------------------------
       QBAR11 = Q11 * DCOS(TH(I)) ** 4 + 2.0 * (Q12+2.0*Q66) *
     &      DSIN(TH(I)) ** 2 * DCOS(TH(I)) ** 2 + Q22 * DSIN(TH(I)) ** 4
       QBAR12 = (Q11+Q22-4.0*Q66) * DSIN(TH(I)) ** 2 * DCOS(TH(I)) ** 2
     &      + Q12 * (DSIN(TH(I))**4+DCOS(TH(I))**4)
       QBAR16 = (Q11-Q12-2.0*Q66) * DSIN(TH(I)) * DCOS(TH(I)) ** 3 + (
     &      Q12-Q22+2.0*Q66) * DSIN(TH(I)) ** 3 * DCOS(TH(I))
       QBAR22 = Q11 * DSIN(TH(I)) ** 4 + 2.0 * (Q12+2.0*Q66) *
     &      DSIN(TH(I)) ** 2 * DCOS(TH(I)) ** 2 + Q22 * DCOS(TH(I)) ** 4
       QBAR26 = (Q11-Q12-2.0*Q66) * DSIN(TH(I)) ** 3 * DCOS(TH(I)) + (
     &      Q12-Q22+2.0*Q66) * DSIN(TH(I)) * DCOS(TH(I)) ** 3
       QBAR44 = Q44 * DCOS(TH(I)) ** 2 + Q55 * DSIN(TH(I)) ** 2
       QBAR45 = (Q44-Q55) * DCOS(TH(I)) * DSIN(TH(I))
       QBAR55 = Q55 * DCOS(TH(I)) ** 2 + Q44 * DSIN(TH(I)) ** 2
       QBAR66 = (Q11+Q22-2.0*Q12-2.0*Q66) * DSIN(TH(I)) ** 2 *
```

```
&    DCOS(TH(I)) ** 2 + Q66 * (DSIN(TH(I))**4+DCOS(TH(I))**4)
c
c-----------------------------------------------------------------
c  TOP AND BOTTOM LOCATION OF PLY(I)
     ZK = ZK1 + TPLY
c  EXTENSIONAL STIFFNESS ELEMENTS
     ZKO = ZK - ZK1
     A11 = QBAR11 * ZKO + A11
     A12 = QBAR12 * ZKO + A12
     A22 = QBAR22 * ZKO + A22
     A16 = QBAR16 * ZKO + A16
     A26 = QBAR26 * ZKO + A26
     A66 = QBAR66 * ZKO + A66
     A44 = QBAR44 * ZKO + A44
     A45 = QBAR45 * ZKO + A45
     A55 = QBAR55 * ZKO + A55
c
c  COUPLING STIFFNESS ELEMENTS
     ZK2 = (ZK**2-ZK1**2) / 2.0
     B11 = QBAR11 * ZK2 + B11
     B12 = QBAR12 * ZK2 + B12
     B22 = QBAR22 * ZK2 + B22
     B16 = QBAR16 * ZK2 + B16
     B26 = QBAR26 * ZK2 + B26
     B66 = QBAR66 * ZK2 + B66
c
c  BENDING STIFFNESS ELEMENTS
     ZK3 = (ZK**3-ZK1**3) / 3.0
     D11 = QBAR11 * ZK3 + D11
     D12 = QBAR12 * ZK3 + D12
     D22 = QBAR22 * ZK3 + D22
     D16 = QBAR16 * ZK3 + D16
     D26 = QBAR26 * ZK3 + D26
     D66 = QBAR66 * ZK3 + D66
     D44 = QBAR44 * ZK3 + D44
     D45 = QBAR45 * ZK3 + D45
     D55 = QBAR55 * ZK3 + D55
c
c  HIGHER ORDER STIFFNESS ELEMENTS
     ZK4 = (ZK**4-ZK1**4) / 4.0
     E11 = QBAR11 * ZK4 + E11
     E12 = QBAR12 * ZK4 + E12
     E22 = QBAR22 * ZK4 + E22
```

```
      E16 = QBAR16 * ZK4 + E16
      E26 = QBAR26 * ZK4 + E26
      E66 = QBAR66 * ZK4 + E66
c
      ZK5 = (ZK**5-ZK1**5) / 5.0
      F11 = QBAR11 * ZK5 + F11
      F12 = QBAR12 * ZK5 + F12
      F22 = QBAR22 * ZK5 + F22
      F16 = QBAR16 * ZK5 + F16
      F26 = QBAR26 * ZK5 + F26
      F66 = QBAR66 * ZK5 + F66
      F44 = QBAR44 * ZK5 + F44
      F45 = QBAR45 * ZK5 + F45
      F55 = QBAR55 * ZK5 + F55
c
      ZK6 = (ZK**6-ZK1**6) / 6.0
      G11 = QBAR11 * ZK6 + G11
      G12 = QBAR12 * ZK6 + G12
      G22 = QBAR22 * ZK6 + G22
      G16 = QBAR16 * ZK6 + G16
      G26 = QBAR26 * ZK6 + G26
      G66 = QBAR66 * ZK6 + G66
c
      ZK7 = (ZK**7-ZK1**7) / 7.0
      H11 = QBAR11 * ZK7 + H11
      H12 = QBAR12 * ZK7 + H12
      H22 = QBAR22 * ZK7 + H22
      H16 = QBAR16 * ZK7 + H16
      H26 = QBAR26 * ZK7 + H26
      H66 = QBAR66 * ZK7 + H66
c
      ZK8 = (ZK**8-ZK1**8) / 8.0
      I11 = QBAR11 * ZK8 + I11
      I12 = QBAR12 * ZK8 + I12
      I22 = QBAR22 * ZK8 + I22
      I16 = QBAR16 * ZK8 + I16
      I26 = QBAR26 * ZK8 + I26
      I66 = QBAR66 * ZK8 + I66
c
      ZK9 = (ZK**9-ZK1**9) / 9.0
      J11 = QBAR11 * ZK9 + J11
      J12 = QBAR12 * ZK9 + J12
      J22 = QBAR22 * ZK9 + J22
```

```
      J16 = QBAR16 * ZK9 + J16
      J26 = QBAR26 * ZK9 + J26
      J66 = QBAR66 * ZK9 + J66
c     GO TO NEXT LAYER
      ZK1 = ZK
   10 Continue
      Return
      End
c-----------------------------------------------------------
      Subroutine GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
     &  D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,
     &  F44,F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,
     &  B11,B12,B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,
     &  G16,G26,G66,I11,I12,I22,I16,I26,I66,NBUCVIB,MMAX,MSIZE,RHO,
     &  STIFF,MASS,BETA,ALPHA,EVAL,EVEC,MSIZESQ,REVEC)
c-----------------------------------------------------------
c     THIS SUBROUTINE GENERATES THE GALERKIN EQUATIONS AND FORMS
c     THE MASS AND STIFFNESS MATRICES. THEN IT CALLS DGVCRG, AN
c     IMSL SUBROUTINE WHICH SOLVES THE EIGENVALUE PROBLEM:
C         [STIFF]{X} = (OMEGA^2 OR N1BAR)[MASS]{X}
C-----------------------------------------------------------
      Double Precision PI, R, H, A, B, A11, A12, A22, A16, A26, A66,
     &  A44, A45, A55, D11, D12, D22, D16, D26, D66, D44, D45, D55,
     &  F11, F12, F22, F16, F26, F66, F44, F45, F55, H11, H12, H22,
     &  H16, H26, H66, J11, J12, J22, J16, J26, J66, B11, B12, B22,
     &  B16, B26, B66, E11, E12, E22, E16, E26, E66, G11, G12, G22,
     &  G16, G26, G66, I11, I12, I22, I16, I26, I66,
     &  STIFF(MSIZE,MSIZE), MASS(MSIZE,MSIZE), AUO, BUO, CUO, EUO,
     &  GUO, AVO, BVO, CVO, EVO, GVO, AW, BW, CW, EW, GW, AJX, BJX,
     &  CJX, EJX, GJX, AJY, BJY, CJY, EJY, GJY
      Double Precision AUOMASS, BUOMASS, CUOMASS, EUOMASS, GUOMASS,          &
     &  AVOMASS, BVOMASS,CVOMASS, EVOMASS, GVOMASS, AWMASS, BWMASS,
     &  CWMASS, EWMASS, GWMASS, AJXMASS, BJXMASS, CJXMASS, EJXMASS,
     &  GJXMASS, AJYMASS, BJYMASS, CJYMASS, EJYMASS, GJYMASS, RHO,
     &  I2BARPR, I3BARPR, I5BAR, I7, I1, I4BAR
      Integer P, Q, M, N, MMAX, NMAX
c     THESE VARIABLES NEEDED FOR THE IMSL EIGENVALUE SOLVER .
      Double Precision BETA(MSIZE), REVAL, OMEGA, AGEVAL, AGEVEC,
     &  REVEC(MSIZESQ)
      Double Complex ALPHA(MSIZE), EVAL(MSIZE), EVEC(MSIZE,MSIZE)
c-----------------------------------------------------------
c     NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS
      NMAX = MMAX
```

```
c   GENERATE GALERKIN EQUATIONS
    I = 1
    J = 1
    Do 10 P = 1, MMAX
     Do 10 Q = 1, NMAX
      Do 20 M = 1, MMAX
       Do 20 N = 1, NMAX
c
c-------------------------------------------------------------
c   COMPUTE STIFFNESS MATRIX ELEMENTS
c-------------------------------------------------------------
c
        If (M.EQ.P.AND.N.EQ.Q) Then
****************************************************************
*    The following equations correspond to the Galerkin     *
*       Equations for Case I, Eqs 67 through 71          *
****************************************************************
c
c       ** corresponding to Eq (67) **
c
            AUO = -(((12*PI**2*B66*A**3*H**2-16*PI**2*E66*A**3)*Q**2
     &      +(12*PI**2*B11*A*B**2*H**2-16*PI**2*E11*A*B**2)*P**2
     &      )*R**2+(24*PI**2*F66*A**3-18*PI**2*D66*A**3*H**2)*Q
     &      **2*R+(6*PI**2*E66*A**3*H**2-8*PI**2*G66*A**3)*Q**2)
     &      / (A**2*B*H**2*R**2) / 48.0
        BUO = -(((12*PI**2*B66+12*PI**2*B12)*A**2*B*H**2+(
     &      -16*PI**2*E66-16*PI**2*E12)*A**2*B)*P*Q*R**2+((
     &      -6*PI**2*D66-12*PI**2*D12)*A**2*B*H**2+(8*PI**2*F66+
     &      16*PI**2*F12)*A**2*B)*P*Q*R) / (A**2*B*H**2*R**2) /
     &      48.0
        CUO = -(((-32*PI**3*E66-16*PI**3*E12)*A**2*P*Q**2-16*PI
     &      **3*E11*B**2*P**3)*R**2+((32*PI**3*F66+16*PI**3*F12)
     &      *A**2*P*Q**2-12*PI*A12*A**2*B**2*H**2*P)*R-8*PI**3*
     &      G66*A**2*P*Q**2) / (A**2*B*H**2*R**2) / 48.0
        EUO = -((12*PI**2*A66*A**3*H**2*Q**2+12*PI**2*A11*A*B**2
     &      *H**2*P**2)*R**2-12*PI**2*B66*A**3*H**2*Q**2*R+3*PI
     &      **2*D66*A**3*H**2*Q**2) / (A**2*B*H**2*R**2) / 48.0
        GUO = -((12*PI**2*A66+12*PI**2*A12)*A**2*B*H**2*P*Q*R**2
     &      -3*PI**2*D66*A**2*B*H**2*P*Q) / (A**2*B*H**2*R**2) /
     &      48.0
c
c       ** corresponding to Eq (68) **
c
```

```
      AVO = -(((12*PI**2*B66+12*PI**2*B12)*A*B**2*H**2+(
 &        -16*PI**2*E66-16*PI**2*E12)*A*B**2)*P*Q*R**2+(8*PI**
 &        2*F66*A*B**2-6*PI**2*D66*A*B**2*H**2)*P*Q*R+(8*PI**2
 &        *G66*A*B**2-6*PI**2*E66*A*B**2*H**2)*P*Q) / (A*B**2*
 &        H**2*R**2) / 48.0
      BVO = -(((12*PI**2*B22*A**2*B*H**2-16*PI**2*E22*A**2*B)*
 &        Q**2+(12*PI**2*B66*B**3*H**2-16*PI**2*E66*B**3)*P**2
 &        )*R**2+((16*PI**2*F22*A**2*B-12*PI**2*D22*A**2*B*H**
 &        2)*Q**2+(6*PI**2*D66*B**3*H**2-8*PI**2*F66*B**3)*P**
 &        2)*R) / (A*B**2*H**2*R**2) / 48.0
      CVO = -((((-32*PI**3*E66-16*PI**3*E12)*B**2*P**2
 &        )*Q-16*PI**3*E22*A**2*Q**3)*R**2+(16*PI**3*F22*A**2*
 &        Q**3-12*PI*A22*A**2*B**2*H**2*Q)*R+8*PI**3*G66*B**2*
 &        P**2*Q) / (48*A*B**2*H**2*R**2)
      EVO = -((12*PI**2*A66+12*PI**2*A12)*A*B**2*H**2*P*Q*R**2
 &        -3*PI**2*D66*A*B**2*H**2*P*Q) / (A*B**2*H**2*R**2) /
 &        48.0
      GVO = -((12*PI**2*A22*A**2*B*H**2*Q**2+12*PI**2*A66*B**3
 &        *H**2*P**2)*R**2+12*PI**2*B66*B**3*H**2*P**2*R+3*PI
 &        **2*D66*B**3*H**2*P**2) / (A*B**2*H**2*R**2) / 48.0
c
c     ** corresponding to Eq (69) **
c
      AW1 = (((24*PI**5*F66+12*PI**5*F12)*A**3*B**2*H**2+(
 &        -32*PI**5*H66-16*PI**5*H12)*A**3*B**2)*P**2*Q**3+((
 &        12*PI**5*F11*A*B**4*H**2-16*PI**5*H11*A*B**4)*P**4+(
 &        -9*PI**3*A55*A**3*B**4*H**4+72*PI**3*D55*A**3*B**4*H
 &        **2-144*PI**3*F55*A**3*B**4)*P**2)*Q) / (PI**2*A**3*
 &        B**3*H**4*P*Q) / 36.0
      AW2 = ((((-36*PI**5*G66-12*PI**5*G12)*A**3*B**2*H**2+(48
 &        *PI**5*I66+16*PI**5*I12)*A**3*B**2)*P**2*Q**3+(9*PI
 &        **3*B12*A**3*B**4*H**4-12*PI**3*E12*A**3*B**4*H**2)*
 &        P**2*Q)*R+(12*PI**5*H66*A**3*B**2*H**2-16*PI**5*J66*
 &        A**3*B**2)*P**2*Q**3) / (PI**2*A**3*B**3*H**4*P*Q*R
 &        **2) / 36.0
      AW = AW1 + AW2
      BW1 = ((12*PI**5*F22*A**4*B*H**2-16*PI**5*H22*A**4*B)*P*
 &        Q**4+(((24*PI**5*F66+12*PI**5*F12)*A**2*B**3*H**2+(
 &        -32*PI**5*H66-16*PI**5*H12)*A**2*B**3)*P**3+(-9*PI**
 &        3*A44*A**4*B**3*H**4+72*PI**3*D44*A**4*B**3*H**2-144
 &        *PI**3*F44*A**4*B**3)*P)*Q**2) / (PI**2*A**3*B**3*H
 &        **4*P*Q) / 36.0
      BW2 = (((32*PI**5*I22*A**4*B-24*PI**5*G22*A**4*B*H**2)*P
```

```
     &       *Q**4+(((-12*PI**5*G66-12*PI**5*G12)*A**2*B**3*H**2+
     &       (16*PI**5*I66+16*PI**5*I12)*A**2*B**3)*P**3+(9*PI**3
     &       *B22*A**4*B**3*H**4-12*PI**3*F22*A**4*B**3*H**2)*P)*
     &       Q**2)*R+(12*PI**5*H22*A**4*B*H**2-16*PI**5*J22*A**4*
     &       B)*P*Q**4+(12*PI**3*F22*A**4*B**3*H**2-9*PI**3*D22*A
     &       **4*B**3*H**4)*P*Q**2) / (PI**2*A**3*B**3*H**4*P*Q*R
     &       **2) / 36.0
      BW = BW1 + BW2
      CW1 = (-16*PI**6*H22*A**4*P*Q**5+((-64*PI**6*H66-32*PI**
     &       6*H12)*A**2*B**2*P**3+(-9*PI**4*A44*A**4*B**2*H**4+
     &       72*PI**4*D44*A**4*B**2*H**2-144*PI**4*F44*A**4*B**2)
     &       *P)*Q**3+((-9*PI**4*A55*A**2*B**4*H**4+72*PI**4*D55*
     &       A**2*B**4*H**2-144*PI**4*F55*A**2*B**4)*P**3-[illegible]PI**
     &       6*H11*B**4*P**5)*Q) / (PI**2*A**3*B**3*H**4*P*Q) /
     &       36.0
      CW2 = ((32*PI**6*I22*A**4*P*Q**5+((64*PI**6*I66+32*PI**6
     &       *I12)*A**2*B**2*P**3-24*PI**4*E22*A**4*B**2*H**2*P)*
     &       Q**3-24*PI**4*E12*A**2*B**4*H**2*P**3*Q)*R-16*PI**6*
     &       J22*A**4*P*Q**5+(24*PI**4*F22*A**4*B**2*H**2*P-16*PI
     &       **6*J66*A**2*B**2*P**3)*Q**3-9*PI**2*A22*A**4*B**4*H
     &       **4*P*Q) / (PI**2*A**3*B**3*H**4*P*Q*R**2) / 36.0
      CW = CW1 + CW2
      EW = (((24*PI**5*E66+12*PI**5*E12)*A**3*B**2*H**2*P**2*Q
     &       **3+12*PI**5*E11*A*B**4*H**2*P**4*Q)*R**2+((-24*PI**
     &       5*F66-12*PI**5*F12)*A**3*B**2*H**2*P**2*Q**3+9*PI**3
     &       *A12*A**3*B**4*H**4*P**2*Q)*R+6*PI**5*G66*A**3*B**2*
     &       H**2*P**2*Q**3) / (PI**2*A**3*B**3*H**4*P*Q*R**2) /
     &       36.0
      GW = ((12*PI**5*E22*A**4*B*H**2*P*Q**4+(24*PI**5*E66+12*
     &       PI**5*E12)*A**2*B**3*H**2*P**3*Q**2)*R**2+(9*PI**3*
     &       A22*A**4*B**3*H**4*P*Q**2-12*PI**5*F22*A**4*B*H**2*P
     &       *Q**4)*R-6*PI**5*G66*A**2*B**3*H**2*P**3*Q**2) / (PI
     &       **2*A**3*B**3*H**4*P*Q*R**2) / 36.0
c
c     ** corresponding to Eq (70) **
c
      AJX = -(((18*PI**2*D66*A**3*H**4-48*PI**2*F66*A**3*H**2+
     &       32*PI**2*H66*A**3)*Q**2+(18*PI**2*D11*A*B**2*H**4-48
     &       *PI**2*F11*A*B**2*H**2+32*PI**2*H11*A*B**2)*P**2
     &       +18*A55*A**3*B**2*H**4-144*D55*A**3*B**2*H**2+288*
     &       F55*A**3*B**2)*R**2+(-36*PI**2*E66*A**3*H**4+96*PI**
     &       2*G66*A**3*H**2-64*PI**2*I66*A**3)*Q**2*R+(18*PI**2*
     &       F66*A**3*H**4-48*PI**2*H66*A**3*H**2+32*PI**2*J66*
```

```
&          A**3)*Q**2) / (72*A**2*B*H**4*R**2)
         BJX = -(((18*PI**2*D66+18*PI**2*D12)*A**2*B*H**4+(
&          -48*PI**2*F66-48*PI**2*F12)*A**2*B*H**2+(32*PI**2*
&          H66+32*PI**2*H12)*A**2*B)*P*Q*R**2+((-18*PI**2*E66-
&          18*PI**2*E12)*A**2*B*H**4+(48*PI**2*G66+48*PI**2*G12
&          )*A**2*B*H**2+(-32*PI**2*I66-32*PI**2*I12)*A**2*B)*P
&          *Q*R) / (A**2*B*H**4*R**2) / 72.0
         CJXA = -((((-48*PI**3*F66-24*PI**3*F12)*A**2*H**2+(64*PI
&          **3*H66+32*PI**3*H12)*A**2)*P*Q**2+(32*PI**3*H11*B**
&          2-24*PI**3*F11*B**2*H**2)*P**3+(18*PI*A55*A**2*B**2*
&          H**4-144*PI*D55*A**2*B**2*H**2+288*PI*F55*A**2*B**2)
&          *P)*R**2+(((72*PI**3*G66+24*PI**3*G12)*A**2*H**2+(
&          -96*PI**3*I66-32*PI**3*I12)*A**2)*P*Q**2+(24*PI*E12
&          *A**2*B**2*H**2-18*PI*B12*A**2*B**2*H**4)*P)*R+(32*
&          PI**3*J66*A**2-24*PI**3*H66*A**2*H**2)*P*Q**2)
         CJX = CJXA / (72*A**2*B*H**4*R**2)
         EJX = -(((18*PI**2*B66*A**3*H**4-24*PI**2*E66*A**3*H**2)
&          *Q**2+(18*PI**2*B11*A*B**2*H**4-24*PI**2*E11*A*B**2*
&          H**2)*P**2)*R**2+(36*PI**2*F66*A**3*H**2-27*PI**2*
&          D66*A**3*H**4)*Q**2*R+(9*PI**2*E66*A**3*H**4-12*PI**
&          2*G66*A**3*H**2)*Q**2) / (A**2*B*H**4*R**2) / 72.0
         GJX = -(((18*PI**2*B66+18*PI**2*B12)*A**2*B*H**4+(
&          -24*PI**2*E66-24*PI**2*E12)*A**2*B*H**2)*P*Q*R**2+(
&          12*PI**2*F66*A**2*B*H**2-9*PI**2*D66*A**2*B*H**4)*P*
&          Q*R+(12*PI**2*G66*A**2*B*H**2-9*PI**2*E66*A**2*B*H**
&          4)*P*Q) / (A**2*B*H**4*R**2) / 72.0
c
c       ** corresponding to Eq (71)
c
         AJY = -(((18*PI**2*D66+18*PI**2*D12)*A*B**2*H**4+(
&          -48*PI**2*F66-48*PI**2*F12)*A*B**2*H**2+(32*PI**2*
&          H66+32*PI**2*H12)*A*B**2)*P*Q*R**2+((-18*PI**2*E66-
&          18*PI**2*E12)*A*B**2*H**4+(48*PI**2*G66+48*PI**2*G12
&          )*A*B**2*H**2+(-32*PI**2*I66-32*PI**2*I12)*A*B**2)*P
&          *Q*R) / (A*B**2*H**4*R**2) / 72.0
         BJY = -(((18*PI**2*D22*A**2*B*H**4-48*PI**2*F22*A**2*B*
&          H**2+32*PI**2*H22*A**2*B)*Q**2+(18*PI**2*D66*B**3*H
&          **4-48*PI**2*F66*B**3*H**2+32*PI**2*H66*B**3)*P**2+
&          18*A44*A**2*B**3*H**4-144*D44*A**2*B**3*H**2+288*
&          F44*A**2*B**3)*R**2+(-36*PI**2*E22*A**2*B*H**4+96*PI
&          **2*G22*A**2*B*H**2-64*PI**2*I22*A**2*B)*Q**2*R+(18*
&          PI**2*F22*A**2*B*H**4-48*PI**2*H22*A**2*B*H**2+32*PI
&          **2*J22*A**2*B)*Q**2) / (A*B**2*H**4*R**2) / 72.0
```

```
      CJY1 = -((32*PI**3*H22*A**2-24*PI**3*F22*A**2*H**2)*Q**3
     &   +(((-48*PI**3*F66-24*PI**3*F12)*B**2*H**2+(64*PI**3*
     &   H66+32*PI**3*H12)*B**2)*P**2++18*PI*A44*A**2*B**2*H
     &   **4-144*PI*D44*A**2*B**2*H**2+288*PI*F44*A**2*B**2)*
     &   Q) / (A*B**2*H**4) / 72.0
      CJY2 = -(((48*PI**3*G22*A**2*H**2-64*PI**3*I22*A**2)*Q**
     &   3+(((24*PI**3*G66+24*PI**3*G12)*B**2*H**2+(-32*PI**3
     &   *I66-32*PI**3*I12)*B**2)*P**2-18*PI*B22*A**2*B**2*H
     &   **4+24*PI*E22*A**2*B**2*H**2)*Q)*R+(32*PI**3*J22*A**
     &   2-24*PI**3*H22*A**2*H**2)*Q**3+(18*PI*D22*A**2*B**2*
     &   H**4-24*PI*F22*A**2*B**2*H**2)*Q) / (A*B**2*H**4*R**
     &   2) / 72.0
      CJY = CJY1 + CJY2
      EJY = -(((18*PI**2*B66+18*PI**2*B12)*A*B**2*H**4+(
     &   -24*PI**2*E66-24*PI**2*E12)*A*B**2*H**2)*P*Q*R**2+((
     &   -9*PI**2*D66-18*PI**2*D12)*A*B**2*H**4+(12*PI**2*F66
     &   +24*PI**2*F12)*A*B**2*H**2)*P*Q*R) / (A*B**2*H**4*R
     &   **2) / 72.0
      GJY = -(((18*PI**2*B22*A**2*B*H**4-24*PI**2*E22*A**2*B*H
     &   **2)*Q**2+(18*PI**2*B66*B**3*H**4-24*PI**2*E66*B**3*
     &   H**2)*P**2)*R**2+((24*PI**2*F22*A**2*B*H**2-18*PI**2
     &   *D22*A**2*B*H**4)*Q**2+(9*PI**2*D66*B**3*H**4-12*PI
     &   **2*F66*B**3*H**2)*P**2)*R) / (A*B**2*H**4*R**2) /
     &   72.0
c
      Else If (MOD(M+P,2).NE.0.AND.MOD(N+Q,2).NE.0) Then
c
*****************************************************************
*   The following equations correspond to the Galerkin       *
*      Equations for Case II, Eqs 72 through 76          *
*****************************************************************
c
c     ** corresponding to Eq (72)
c
      AUO = ((24*PI*B16*A*B**2*H**2-32*PI*E16*A*B**2)*M**2*N*
     &   Q*R**2+(24*PI*F16*A*B**2-18*PI*D16*A*B**2*H**2)*M**
     &   2*N*Q*R) / (((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**2
     &   *M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H
     &   **2*M**2*N**2)*R**2)
      BUO = (((12*PI*B26*A**2*B*H**2-16*PI*E26*A**2*B)*M*N**2+
     &   (12*PI*B16*B**3*H**2-16*PI*E16*B**3)*M**3)*Q*R**2+(
     &   24*PI*F26*A**2*B-18*PI*D26*A**2*B*H**2)*M*N**2*Q*R+(
     &   6*PI*E26*A**2*B*H**2-8*PI*G26*A**2*B)*M*N**2*Q) / ((
```

```
     &       (3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**2*M**2)*Q**2-3
     &       *PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M**2*N**2
     &       )*R**2)
      CUO = ((-16*PI**2*E26*A**2*M*N**3-48*PI**2*E16*B**2*M**3
     &       *N)*Q*R**2+(24*PI**2*F26*A**2*M*N**3+(24*PI**2*F16*B
     &       **2*M**3-12*A26*A**2*B**2*H**2*M)*N)*Q*R+(6*B26*A**2
     &       *B**2*H**2*M*N-8*PI**2*G26*A**2*M*N**3)*Q) / (((3*PI
     &       *A*B**2*H**2*P**2-3*PI*A*B**2*H**2*M**2)*Q**2-3*PI*A
     &       *B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M**2*N**2)*R**
     &       2)
      EUO = (24*PI*A16*A*B**2*H**2*M**2*N*Q*R**2-12*PI*B16*A*B
     &       **2*H**2*M**2*N*Q*R) / (((3*PI*A*B**2*H**2*P**2-3*PI
     &       *A*B**2*H**2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3
     &       *PI*A*B**2*H**2*M**2*N**2)*R**2)
      GUO = ((12*PI*A26*A**2*B*H**2*M*N**2+12*PI*A16*B**3*H**2
     &       *M**3)*Q*R**2+(6*PI*B16*B**3*H**2*M**3-6*PI*B26*A**2
     &       *B*H**2*M*N**2)*Q*R) / (((3*PI*A*B**2*H**2*P**2-3*PI
     &       *A*B**2*H**2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3
     &       *PI*A*B**2*H**2*M**2*N**2)*R**2)
c
c     ** corresponding to Eq (73) **
c
      AVO = (((12*PI*B26*A**3*H**2-16*PI*E26*A**3)*N**3+(12*PI
     &       *B16*A*B**2*H**2-16*PI*E16*A*B**2)*M**2*N)*P*R**2+((
     &       16*PI*F26*A**3-12*PI*D26*A**3*H**2)*N**3+(6*PI*D16*A
     &       *B**2*H**2-8*PI*F16*A*B**2)*M**2*N)*P*R) / (((3*PI*A
     &       **2*B*H**2*P**2-3*PI*A**2*B*H**2*M**2)*Q**2-3*PI*A**
     &       2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**2*N**2)*R**2)
      BVO = ((24*PI*B26*A**2*B*H**2-32*PI*E26*A**2*B)*M*N**2*P
     &       *R**2+(8*PI*F26*A**2*B-6*PI*D26*A**2*B*H**2)*M*N**2*
     &       P*R+(8*PI*G26*A**2*B-6*PI*E26*A**2*B*H**2)*M*N**2*P)
     &       / (((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H**2*M**2)*Q
     &       **2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**2
     &       *N**2)*R**2)
      CVO = ((-48*PI**2*E26*A**2*M*N**3-16*PI**2*E16*B**2*M**3
     &       *N)*P*R**2+(24*PI**2*F26*A**2*M*N**3+(-8*PI**2*F16*B
     &       **2*M**3-12*A26*A**2*B**2*H**2*M)*N)*P*R+(8*PI**2*
     &       G26*A**2*M*N**3-6*B26*A**2*B**2*H**2*M*N)*P) / (((3*
     &       PI*A**2*B*H**2*P**2-3*PI*A**2*B*H**2*M**2)*Q**2-3*PI
     &       *A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**2*N**2)*R
     &       **2)
      EVO = ((12*PI*A26*A**3*H**2*N**3+12*PI*A16*A*B**2*H**2*M
     &       **2*N)*P*R**2+(6*PI*B16*A*B**2*H**2*M**2*N-6*PI*B26*
```

```
&         A**3*H**2*N**3)*P*R) / (((3*PI*A**2*B*H**2*P**2-3*PI
&         *A**2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3
&         *PI*A**2*B*H**2*M**2*N**2)*R**2)
      GVO = (24*PI*A26*A**2*B*H**2*M*N**2*P*R**2+12*PI*B26*A**
&         2*B*H**2*M*N**2*P*R) / (((3*PI*A**2*B*H**2*P**2-3*PI
&         *A**2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3
&         *PI*A**2*B*H**2*M**2*N**2)*R**2)
c
c     ** corresponding to Eq (74) **
c
      AW1 = -((48*PI**2*F26*A**3*H**2-64*PI**2*H26*A**3)*N**3+
&         ((144*PI**2*F16*A*B**2*H**2-192*PI**2*H16*A*B**2)*M
&         **2-36*A45*A**3*B**2*H**4+288*D45*A**3*B**2*H**2-576
&         *F45*A**3*B**2)*N) * P * Q / ((9*PI*A**2*B**2*H**4*P
&         **2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*A**2*B**2*H
&         **4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)
      AW2 = -(((128*PI**2*I26*A**3-96*PI**2*G26*A**3*H**2)*N**
&         3+((128*PI**2*I16*A*B**2-96*PI**2*G16*A*B**2*H**2)*M
&         **2+36*B26*A**3*B**2*H**4-48*E26*A**3*B**2*H**2)*N)*
&         P*Q*R+((48*PI**2*H26*A**3*H**2-64*PI**2*J26*A**3)*N
&         **3+(48*F26*A**3*B**2*H**2-36*D26*A**3*B**2*H**4)*N)
&         *P*Q) / (((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H
&         **4*M**2)*Q**2-9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A
&         **2*B**2*H**4*M**2*N**2)*R**2)
      AW = AW1 + AW2
      BW1 = -((144*PI**2*F26*A**2*B*H**2-192*PI**2*H26*A**2*B)
&         *M*N**2+(48*PI**2*F16*B**3*H**2-64*PI**2*H16*B**3)*M
&         **3+(-36*A45*A**2*B**3*H**4+288*D45*A**2*B**3*H**2-
&         576*F45*A**2*B**3)*M) * P * Q / ((9*PI*A**2*B**2*H**
&         4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*A**2*B**2
&         *H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)
      BW2 =-(((256*PI**2*I26*A**2*B-192*PI**2*G26*A**2*B*H**2)
&         *M*N**2+(36*B26*A**2*B**3*H**4-48*E26*A**2*B**3*H**2
&         )*M)*P*Q*R+(48*PI**2*H26*A**2*B*H**2-64*PI**2*J26*A
&         **2*B)*M*N**2*P*Q) / (((9*PI*A**2*B**2*H**4*P**2-9*
&         PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*A**2*B**2*H**4*N**
&         2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)*R**2)
      BW = BW1 + BW2
      CW = -((((-72*PI*A45*A**2*B**2*H**4+576*PI*D45*A**2*B**2
&         *H**2-1152*PI*F45*A**2*B**2)*M-256*PI**3*H16*B**2*M
&         **3)*N-256*PI**3*H26*A**2*M*N**3)*P*Q*R**2+(384*PI**
&         3*I26*A**2*M*N**3+(128*PI**3*I16*B**2*M**3-192*PI*
&         E26*A**2*B**2*H**2*M)*N)*P*Q*R+(96*PI*F26*A**2*B**2*
```

```
     &      H**2*M*N-128*PI**3*J26*A**2*M*N**3)*P*Q) / (((9*PI*A
     &      **2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*
     &      PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2
     &      *N**2)*R**2)
      EW = -((48*PI**2*E26*A**3*H**2*N**3+144*PI**2*E16*A*B**2
     &      *H**2*M**2*N)*P*Q*R**2+((36*A26*A**3*B**2*H**4-72*PI
     &      **2*F16*A*B**2*H**2*M**2)*N-72*PI**2*F26*A**3*H**2*N
     &      **3)*P*Q*R+(24*PI**2*G26*A**3*H**2*N**3-18*B26*A**3*
     &      B**2*H**4*N)*P*Q) / (((9*PI*A**2*B**2*H**4*P**2-9*PI
     &      *A**2*B**2*H**4*M**2)*Q**2-9*PI*A**2*B**2*H**4*N**2*
     &      P**2+9*PI*A**2*B**2*H**4*M**2*N**2)*R**2)
      GW = -((144*PI**2*E26*A**2*B*H**2*M*N**2+48*PI**2*E16*B
     &      **3*H**2*M**3)*P*Q*R**2+(-72*PI**2*F26*A**2*B*H**2*M
     &      *N**2+24*PI**2*F16*B**3*H**2*M**3+36*A26*A**2*B**3*H
     &      **4*M)*P*Q*R+(18*B26*A**2*B**3*H**4*M-24*PI**2*G26*A
     &      **2*B*H**2*M*N**2)*P*Q) / (((9*PI*A**2*B**2*H**4*P**
     &      2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*A**2*B**2*H**4
     &      *N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)*R**2)
c
c     ** corresponding to Eq (75) **
c
      AJX = ((72*PI**2*D16*A*B**2*H**4-192*PI**2*F16*A*B**2*H**
     &      2+128*PI**2*H16*A*B**2)*M**2*N*Q*R**2+(-72*PI**2*E1
     &      6*AB**2*H**4+192*PI**2*G16*A*B**2*H**2-128*PI**2*I1
     &      6*A*B**2)*M**2*N*Q*R)/(((9*PI**2*A*B**2*H**4*P**2-9
     &      *PI**2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B**2*H**4*N
     &      **2*P**2+9*PI**2*A*B**2*H**4*M**2*N**2)*R**2)
c
      BJX1 = ((36*PI**2*D26*A**2*B*H**4-96*PI**2*F26*A**2*B*H
     &      **2+64*PI**2*H26*A**2*B)*M*N**2+(36*PI**2*D16*B**3*H
     &      **4-96*PI**2*F16*B**3*H**2+64*PI**2*H16*B**3)*M**3+(
     &      36*A45*A**2*B**3*H**4-288*D45*A**2*B**3*H**2+576*F45
     &      *A**2*B**3)*M) * Q / ((9*PI**2*A*B**2*H**4*P**2-9*PI
     &      **2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B**2*H**4*N**2*
     &      P**2+9*PI**2*A*B**2*H**4*M**2*N**2)
      BJX2 = ((-72*PI**2*E26*A**2*B*H**4+192*PI**2*G26*A**2*B*
     &      H**2-128*PI**2*I26*A**2*B)*M*N**2*Q*R+(36*PI**2*F26*
     &      A**2*B*H**4-96*PI**2*H26*A**2*B*H**2+64*PI**2*J26*A
     &      **2*B)*M*N**2*Q) / (((9*PI**2*A*B**2*H**4*P**2-9*PI
     &      **2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B**2*H**4*N**2*
     &      P**2+9*PI**2*A*B**2*H**4*M**2*N**2)*R**2)
      BJX = BJX1 + BJX2
      CJX1 = ((64*PI**3*H26*A**2-48*PI**3*F26*A**2*H**2)*M*N**
```

```
     &    3+((192*PI**3*H16*B**2-144*PI**3*F16*B**2*H**2)*M**3
     &    +(36*PI*A45*A**2*B**2*H**4-288*PI*D45*A**2*B**2*H**2
     &    +576*PI*F45*A**2*B**2)*M)*N) * Q / ((9*PI**2*A*B**2*
     &    H**4*P**2-9*PI**2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B
     &    **2*H**4*N**2*P**2+9*PI**2*A*B**2*H**4*M**2*N**2)
      CJX2 = (((96*PI**3*G26*A**2*H**2-128*PI**3*I26*A**2)*M*
     &    N**3+((96*PI**3*G16*B**2*H**2-128*PI**3*I16*B**2)*M
     &    **3+(48*PI*E26*A**2*B**2*H**2-36*PI*B26*A**2*B**2*H
     &    **4)*M)*N)*Q*R+((64*PI**3*J26*A**2-48*PI**3*H26*A**2
     &    *H**2)*M*N**3+(36*PI*D26*A**2*B**2*H**4-48*PI*F26*A
     &    **2*B**2*H**2)*M*N)*Q) / (((9*PI**2*A*B**2*H**4*P**2
     &    -9*PI**2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B**2*H**4*
     &    N**2*P**2+9*PI**2*A*B**2*H**4*M**2*N**2)*R**2)
      CJX = CJX1 + CJX2
      EJX = ((72*PI**2*B16*A*B**2*H**4-96*PI**2*E16*A*B**2*H**
     &    2)*M**2*N*Q*R**2+(72*PI**2*F16*A*B**2*H**2-54*PI**2*
     &    D16*A*B**2*H**4)*M**2*N*Q*R) / (((9*PI**2*A*B**2*H**
     &    4*P**2-9*PI**2*A*B**2*H**4*M**2)*Q**2-9*PI**2*A*B**2
     &    *H**4*N**2*P**2+9*PI**2*A*B**2*H**4*M**2*N**2)*R**2)
      GJX = (((36*PI**2*B26*A**2*B*H**4-48*PI**2*E26*A**2*B*H
     &    **2)*M*N**2+(36*PI**2*B16*B**3*H**4-48*PI**2*E16*B**
     &    3*H**2)*M**3)*Q*R**2+((48*PI**2*F26*A**2*B*H**2-36*
     &    PI**2*D26*A**2*B*H**4)*M*N**2+(18*PI**2*D16*B**3*H**
     &    4-24*PI**2*F16*B**3*H**2)*M**3)*Q*R) / (((9*PI**2*A*
     &    B**2*H**4*P**2-9*PI**2*A*B**2*H**4*M**2)*Q**2-9*PI**
     &    2*A*B**2*H**4*N**2*P**2+9*PI**2*A*B**2*H**4*M**2*N**
     &    2)*R**2)
c
c     ** corresponding to Eq (76)
c
      AJY1 = ((36*PI**2*D26*A**3*H**4-96*PI**2*F26*A**3*H**2+
     &    64*PI**2*H26*A**3)*N**3+((36*PI**2*D16*A*B**2*H**4-
     &    96*PI**2*F16*A*B**2*H**2+64*PI**2*H16*A*B**2)*M**2+
     &    36*A45*A**3*B**2*H**4-288*D45*A**3*B**2*H**2+576*F45
     &    *A**3*B**2)*N) * P / ((9*PI**2*A**2*B*H**4*P**2-9*PI
     &    **2*A**2*B*H**4*M**2)*Q**2-9*PI**2*A**2*B*H**4*N**2*
     &    P**2+9*PI**2*A**2*B*H**4*M**2*N**2)
      AJY2 = ((-72*PI**2*E26*A**3*H**4+192*PI**2*G26*A**3*H**2
     &    -128*PI**2*I26*A**3)*N**3*P*R+(36*PI**2*F26*A**3*H**
     &    4-96*PI**2*H26*A**3*H**2+64*PI**2*J26*A**3)*N**3*P)
     &    / (((9*PI**2*A**2*B*H**4*P**2-9*PI**2*A**2*B*H**4*M
     &    **2)*Q**2-9*PI**2*A**2*B*H**4*N**2*P**2+9*PI**2*A**2
     &    *B*H**4*M**2*N**2)*R**2)
```

```
      AJY = AJY1 + AJY2
      BJY = ((72*PI**2*D26*A**2*B*H**4-192*PI**2*F26*A**2*B*H
     &   **2+128*PI**2*H26*A**2*B)*M*N**2*P*R**2+(-72*PI**2*
     &   E26*A**2*B*H**4+192*PI**2*G26*A**2*B*H**2-128*PI**2*
     &   I26*A**2*B)*M*N**2*P*R) / (((9*PI**2*A**2*B*H**4*P**
     &   2-9*PI**2*A**2*B*H**4*M**2)*Q**2-9*PI**2*A**2*B*H**4
     &   *N**2*P**2+9*PI**2*A**2*B*H**4*M**2*N**2)*R**2)
      CJY1 = ((192*PI**3*H26*A**2-144*PI**3*F26*A**2*H**2)*M*N
     &   **3+((64*PI**3*H16*B**2-48*PI**3*F16*B**2*H**2)*M**3
     &   +(36*PI*A45*A**2*B**2*H**4-288*PI*D45*A**2*B**2*H**2
     &   +576*PI*F45*A**2*B**2)*M)*N) * P / ((9*PI**2*A**2*B*
     &   H**4*P**2-9*PI**2*A**2*B*H**4*M**2)*Q**2-9*PI**2*A**
     &   2*B*H**4*N**2*P**2+9*PI**2*A**2*B*H**4*M**2*N**2)
      CJY2 = (((192*PI**3*G26*A**2*H**2-256*PI**3*I26*A**2)*M*
     &   N**3+(48*PI*E26*A**2*B**2*H**2-36*PI*B26*A**2*B**2*H
     &   **4)*M*N)*P*R+(64*PI**3*J26*A**2-48*PI**3*H26*A**2*H
     &   **2)*M*N**3*P) / (((9*PI**2*A**2*B*H**4*P**2-9*PI**2
     &   *A**2*B*H**4*M**2)*Q**2-9*PI**2*A**2*B*H**4*N**2*P**
     &   2+9*PI**2*A**2*B*H**4*M**2*N**2)*R**2)
      CJY = CJY1 + CJY2
      EJY = (((36*PI**2*B26*A**3*H**4-48*PI**2*E26*A**3*H**2)*
     &   N**3+(36*PI**2*B16*A*B**2*H**4-48*PI**2*E16*A*B**2*H
     &   **2)*M**2*N)*P*R**2+(72*PI**2*F26*A**3*H**2-54*PI**2
     &   *D26*A**3*H**4)*N**3*P*R+(18*PI**2*E26*A**3*H**4-24*
     &   PI**2*G26*A**3*H**2)*N**3*P) / (((9*PI**2*A**2*B*H**
     &   4*P**2-9*PI**2*A**2*B*H**4*M**2)*Q**2-9*PI**2*A**2*B
     &   *H**4*N**2*P**2+9*PI**2*A**2*B*H**4*M**2*N**2)*R**2)
      GJY = ((72*PI**2*B26*A**2*B*H**4-96*PI**2*E26*A**2*B*H**
     &   2)*M*N**2*P*R**2+(24*PI**2*F26*A**2*B*H**2-18*PI**2*
     &   D26*A**2*B*H**4)*M*N**2*P*R+(24*PI**2*G26*A**2*B*H**
     &   2-18*PI**2*E26*A**2*B*H**4)*M*N**2*P) / (((9*PI**2*A
     &   **2*B*H**4*P**2-9*PI**2*A**2*B*H**4*M**2)*Q**2-9*PI
     &   **2*A**2*B*H**4*N**2*P**2+9*PI**2*A**2*B*H**4*M**2*N
     &   **2)*R**2)
c
     Else
c
c
      AUO = 0.0
      BUO = 0.0
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
```

```
        AVO = 0.0
        BVO = 0.0
        CVO = 0.0
        EVO = 0.0
        GVO = 0.0
        AW = 0.0
        BW = 0.0
        CW = 0.0
        EW = 0.0
        GW = 0.0
        AJX = 0.0
        BJX = 0.0
        CJX = 0.0
        EJX = 0.0
        GJX = 0.0
        AJY = 0.0
        BJY = 0.0
        CJY = 0.0
        EJY = 0.0
        GJY = 0.0
      End If
c
C
c----------------------------------------------
c   STORE THESE TERMS IN THE STIFFNESS MATRIX
c----------------------------------------------
      STIFF(I,J) = AUO
      STIFF(I,J+MMAX*NMAX) = BUO
      STIFF(I,J+2*MMAX*NMAX) = CUO
      STIFF(I,J+3*MMAX*NMAX) = EUO
      STIFF(I,J+4*MMAX*NMAX) = GUO
      STIFF(I+MMAX*NMAX,J) = AVO
      STIFF(I+MMAX*NMAX,J+MMAX*NMAX) = BVO
      STIFF(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVO
      STIFF(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVO
      STIFF(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVO
      STIFF(I+2*MMAX*NMAX,J) = AW
      STIFF(I+2*MMAX*NMAX,J+MMAX*NMAX) = BW
      STIFF(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CW
      STIFF(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EW
      STIFF(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GW
      STIFF(I+3*MMAX*NMAX,J) = AJX
      STIFF(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJX
```

```
      STIFF(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJX
      STIFF(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJX
      STIFF(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJX
      STIFF(I+4*MMAX*NMAX,J) = AJY
      STIFF(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJY
      STIFF(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJY
      STIFF(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJY
      STIFF(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJY
c---------------------------------------------------------------
C   COMPUTE MASS MATRIX ELEMENTS
c---------------------------------------------------------------
c   FIRST CALCULATE THE MASS MOMENTS OF INERTIA.
      I2BARPR = RHO * H ** 3 / (15.0*R)
          I3BARPR = RHO * H ** 3 / (60.0*R)
          I5BAR = RHO * H ** 3 * 4.0 / 315.0
          I7 = RHO * H ** 7 / 448.0
          I1 = RHO * H
          I4BAR = RHO * H ** 3 * 17.0 / 315.0
          AUOMASS = 0.0
      BUOMASS = 0.0
      CUOMASS = 0.0
      EUOMASS = 0.0
      GUOMASS = 0.0
      AVOMASS = 0.0
      EVOMASS = 0.0
      GVOMASS = 0.0
      EWMASS = 0.0
      GWMASS = 0.0
      BJXMASS = 0.0
      EJXMASS = 0.0
      GJXMASS = 0.0
      AJYMASS = 0.0
      EJYMASS = 0.0
      GJYMASS = 0.0
c
      If (M.EQ.P.AND.N.EQ.Q) Then
c
       If (NBUCVIB.EQ.1) Then
c   VIBRATIONS PROBLEM - WE ARE LOOKING FOR THE NATURAL
c    FREQUENCIES
c
      BVOMASS = -A * B * I2BARPR / 4.0
            CVOMASS = PI * A * I3BARPR * Q / 4.0
```

```
            AWMASS = PI * B * I5BAR * P / 4.0
            BWMASS = PI * A * I5BAR * Q / 4.0
            CWMASS = -(16*PI**2*A**2*I7*Q**2+16*PI**2*B**2*I7*P**2
     &          +9*A**2*B**2*H**4*I1) / (A*B*H**4) / 36.0
            AJXMASS = -A * B * I4BAR / 4.0
            CJXMASS = PI * B * I5BAR * P / 4.0
            BJYMASS = -A * B * I4BAR / 4.0
            CJYMASS = PI * A * I5BAR * Q / 4.0
c
       ELSE
c
c      BUCKLING PROBLEM - WE ARE LOOKING FOR THE CRITICAL BUCKLING
c       LOADS
c
         BVOMASS = 0.0
         CVOMASS = 0.0
         AWMASS = 0.0
         BWMASS = 0.0
         CWMASS = PI ** 2 * B * P ** 2 / A / 4.0
            AJXMASS = 0.0
         CJXMASS = 0.0
         BJYMASS = 0.0
         CJYMASS = 0.0
c
         End If
c
         Else
c
         BVOMASS = 0.0
         CVOMASS = 0.0
         AWMASS = 0.0
         BWMASS = 0.0
         CWMASS = 0.0
         AJXMASS = 0.0
         CJXMASS = 0.0
       . BJYMASS = 0.0
         CJYMASS = 0.0
c
c
         End If
c-------------------------------------------------------------
c     STORE THESE TERMS IN THE MASS MATRIX
c-------------------------------------------------------------
```

```
      MASS(I,J) = AUOMASS
      MASS(I,J+MMAX*NMAX) = BUOMASS
      MASS(I,J+2*MMAX*NMAX) = CUOMASS
      MASS(I,J+3*MMAX*NMAX) = EUOMASS
      MASS(I,J+4*MMAX*NMAX) = GUOMASS
      MASS(I+MMAX*NMAX,J) = AVOMASS
      MASS(I+MMAX*NMAX,J+MMAX*NMAX) = BVOMASS
      MASS(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVOMASS
      MASS(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVOMASS
      MASS(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVOMASS
      MASS(I+2*MMAX*NMAX,J) = AWMASS
      MASS(I+2*MMAX*NMAX,J+MMAX*NMAX) = BWMASS
      MASS(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CWMASS
      MASS(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EWMASS
      MASS(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GWMASS
      MASS(I+3*MMAX*NMAX,J) = AJXMASS
      MASS(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJXMASS
      MASS(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJXMASS
      MASS(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJXMASS
      MASS(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJXMASS
      MASS(I+4*MMAX*NMAX,J) = AJYMASS
      MASS(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJYMASS
      MASS(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJYMASS
      MASS(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJYMASS
      MASS(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJYMASS
c-----------------------------------------------------------
      J = J + 1
 20   Continue
      I = I + 1
      J = 1
 10 Continue
c-----------------------------------------------------------
c   CALL THE IMSL LIBRARY SUBROUTINE. USE THE MASS AND STIFFNESS
c   MATRICES AS INPUT AND FIND THE EIGENVALUES AND EIGENVECTORS.
    Call DGVCRG(MSIZE,STIFF,MSIZE,MASS,MSIZE,ALPHA,BETA,EVEC,MSIZE)
    Do 40 I = 1, MSIZE
     If (BETA(I).NE.0.0) Then
      EVAL(I) = ALPHA(I) / BETA(I)
        Else
      EVAL(I) = (1.0D+30,0.0D+00)
        End If
 40 Continue
    If (NBUCVIB.EQ.1) Then
```

```
c   PRINT OUT THE FIRST 10 MODES FOR THE VIBRATION PROBLEM
c
       Do 50 I = 1, 10
        REVAL = DREAL(EVAL(I))
        AGEVAL = DIMAG(EVAL(I))
        If (ABS(AGEVAL).GT.1.0D-15) Then
         Write (2,115) I
        Else If (REVAL.GT.1.0D+28) Then
         Write (2,125) I
        Else If (REVAL.LT.0.0) Then
         Write (2,120) I
        Else
         OMEGA = SQRT(REVAL)
         Write(2,*)  'EIGENVALUE POSITIVE REAL'
            Write (2,130) I, REVAL, OMEGA
        End If
  50   Continue
c
      Else
c
c   PRINT OUT THE CRITICAL BUCKLING LOAD. THE CRITICAL
c   BUCKLING LOAD IS THE EIGENVALUE WITH THE SMALLEST ABSOLUTE
c   VALUE .
C
       Do 55 I = 2, MSIZE
        If (ABS(DIMAG(EVAL(I-1))).GT.1.0D-15) Then
         Goto 55
        End If
        If (ABS(DREAL(EVAL(I))).GT.ABS(DREAL(EVAL(I-1))).AND.
     &      ABS(DREAL(EVAL(I-1))).LT.1.0D+28) Then
         Write (2,220) DREAL(EVAL(I-1))
        End If
  55   Continue
c
      End If
c
c   PRINT OUT THE 1ST MODE OF THE DEFLECTION, W(X,Y), ALONG THE
c   MIDLINES OF THE PANEL: X = A/2 AND Y = B/2
c
c   PRINT OUT THE W EIGENVECTOR, CMN
c
      II = 1
      Write (2,500)
```

```
      Write (2,510)
      MNWMIN = 1 + 2 * MMAX * NMAX
      MNWMAX = 3 * MMAX * NMAX
      Do 400 I = MNWMIN, MNWMAX
       REVEC(II) = DREAL(EVEC(I,1))
       AGEVEC = DIMAG(EVEC(I,1))
       If (ABS(AGEVEC).GT.1.0D-15) Then
        Write (2,520) I, II, REVEC(II)
       Else
        Write (2,530) I, II, REVEC(II)
       End If
       II = II + 1
  400 Continue
c
c    DETERMINE W(X=A/2,Y)
c
      ASTEP = A / 50.0
      BSTEP = B / 50.0
      XCOORD = A / 2.0
      YCOORD = 0.0
      Write (2,540)
      Write (2,542)
  801 WMODE = 0.0
      JJJ = 1
      Do 470 M = 1, MMAX
       Do 472 N = 1, NMAX
        WMODE = WMODE + REVEC(JJJ) * SIN(M*PI*XCOORD/A) *
     &      SIN(N*PI*YCOORD/B)
        JJJ = JJJ + 1
  472  Continue
  470 Continue
      Write (2,550) YCOORD, WMODE
      YCOORD = YCOORD + BSTEP
      If (YCOORD.GT.B) Then
       Goto 800
      Else
       Goto 801
      End If
c
  800 YCOORD = B / 2.0
c
c    DETERMINE W(X, Y=B/2)
c
```

```
      XCOORD = 0.0
      Write (2,560)
      Write (2,570)
  810 WMODE = 0.0
      JJJ = 1
      Do 480 M = 1, MMAX
       Do 482 N = 1, NMAX
        WMODE = WMODE + REVEC(JJJ) * SIN(M*PI*XCOORD/A) *
     &      SIN(N*PI*YCOORD/B)
        JJJ = JJJ + 1
  482  Continue
  480 Continue
      Write (2,550) XCOORD, WMODE
      XCOORD = XCOORD + ASTEP
      If (XCOORD.GT.A) Then
       Goto 850
      Else
       Goto 810
      End If
c--------------------------------------------------------------
  115 Format (/,8X,I3,11X,'EIGENVALUE IS COMPLEX')
  120 Format (/,9X,I3,11X,'EIGENVALUE IS NEGATIVE')
  125 Format (/,9X,I3,11X,'EIGENVALUE IS INFINITE')
  130 Format (/,9X,I3,10X,D20.13,12X,D20.13)
  200 Format (/,8X,I3,10X,D20.13)
  220 Format (//,5X,'CRITICAL BUCKLING LOAD = ',1X,D20.13)
  500 Format (//,5X,'W EIGENVECTOR, CMN, FOR 1ST MODE')
  510 Format (//,5X,'M, N',10X,'CMN')
  520 Format (/,5X,I4,2X,I4,12X,D20.13,3X,'COMPLEX')
  530 Format (/,5X,I4,2X,I4,12X,D20.13)
  540 Format (//,5X,'DEFLECTION, W(X=A/2,Y)')
  542 Format (//,5X,'Y(IN.)',10X,'W(A/2,Y)(IN.)')
  550 Format (/,5X,F6.2,11X,E15.8)
  560 Format (//,5X,'DEFLECTION, W(X,Y=B/2)')
  570 Format (//,5X,'X(IN.)',10X,'W(X, B/2)(IN.)')
  850 Return
      End
```

The following is the subroutine GALERK for the clamped boundary condition.

```
c-----------------------------------------------------------
   Subroutine GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
  & D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,
  & F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,
  & B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,
  & I11,I12,I22,I16,I26,I66,NBUCVIB,MMAX,MSIZE,RHO,STIFF,MASS,BETA,
  & ALPHA,EVAL,EVEC,MSIZESQ,REVEC)
c-----------------------------------------------------------
c  THIS SUBROUTINE GENERATES THE GALERKIN EQUATIONS AND FORMS
c  THE MASS AND STIFFNESS MATRICES. THEN IT CALLS DGVCRG, AN
c  IMSL SUBROUTINE WHICH SOLVES THE EIGENVALUE PROBLEM:
c
c     [STIFF]{X} = (OMEGA^2 OR N1BAR)[MASS]{X}
c-----------------------------------------------------------
   Double Precision PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
  & D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,
  & F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,
  & B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,
  & I11,I12,I22,I16,I26,I66,STIFF(MSIZE,MSIZE),MASS(MSIZE,MSIZE),
  & AUO,BUO,CUO,EUO,GUO,AVO,BVO,CVO,EVO,GVO,AW,BW,CW,EW,GW,AJX,BJX,
  & CJX,EJX,GJX,AJY,BJY,CJY,EJY,GJY,AUOMASS,BUOMASS,CUOMASS,EUOMASS,
  & GUOMASS,AVOMASS,BVOMASS
   Double Precision CVOMASS,EVOMASS,GVOMASS,AWMASS,BWMASS,CWMASS,
  & EWMASS,GWMASS,AJXMASS,BJXMASS,CJXMASS,EJXMASS,GJXMASS,AJYMASS,
  & BJYMASS,CJYMASS,EJYMASS,GJYMASS,RHO,I2BARPR,I3BARPR,I5BAR,I7,I1,
  & I4BAR
   Integer P,Q,M,N,MMAX,NMAX
c  THESE VARIABLES NEEDED FOR THE IMSL EIGENVALUE SOLVER .
   Double Precision BETA(MSIZE),REVAL,OMEGA,AGEVAL,AGEVEC,
  & REVEC(MSIZESQ)
   Double Complex ALPHA(MSIZE),EVAL(MSIZE),EVEC(MSIZE,MSIZE)
c-----------------------------------------------------------
c  NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS
   NMAX = MMAX
c  GENERATE GALERKIN EQUATIONS
   I = 1
   J = 1
   Do 10 P = 1,MMAX
    Do 10 Q = 1,MMAX
     Do 20 M = 1,MMAX
```

```
      Do 20 N = 1,MMAX
c
c-------------------------------------------------------------
c   COMPUTE STIFFNESS MATRIX ELEMENTS
c-------------------------------------------------------------
*****************************************************************
*    The following equations correspond to the Galerkin       *
*              Equations for Case I                   *
*****************************************************************
c
       If (M.EQ.P.AND.N.EQ.Q) Then
c
c          ** corresponding to u_o **
c
       AUO = 0.0
       BUO = 0.0
       CUO = -(((-32*PI**3*E66-16*PI**3*E12)*A**2*P*Q**2-16*PI**3*
    &     E11*B**2*P**3)*R**2+((32*PI**3*F66+16*PI**3*F12)*A**2*P*Q**
    &     2-12*PI*A12*A**2*B**2*H**2*P)*R-8*PI**3*G66*A**2*P*Q**2)/(A
    &     **2*B*H**2*R**2)/48.0
c
       EUO = -((12*PI**2*A66*A**3*H**2*Q**2+12*PI**2*A11*A*B**2*H**2
    &     *P**2)*R**2-12*PI**2*B66*A**3*H**2*Q**2*R+3*PI**2*D66*A**3*
    &     H**2*Q**2)/(A**2*B*H**2*R**2)/48.0
c
       GUO = -((12*PI**2*A66+12*PI**2*A12)*A**2*B*H**2*P*Q*R**2-3*PI
    &     **2*D66*A**2*B*H**2*P*Q)/(A**2*B*H**2*R**2)/48.0
c
c          ** corresponding to v_o **
c
       AVO = 0.0
       BVO = 0.0
c
       CVO = -(((-32*PI**3*E66-16*PI**3*E12)*B**2*P**2*Q-16*PI**3*
    &     E22*A**2*Q**3)*R**2+(16*PI**3*F22*A**2*Q**3-12*PI*A22*A**2*
    &     B**2*H**2*Q)*R+8*PI**3*G66*B**2*P**2*Q)/(A*B**2*H**2*R**2)/
    &     48.0
c
       EVO = -((12*PI**2*A66+12*PI**2*A12)*A*B**2*H**2*P*Q*R**2-3*PI
    &     **2*D66*A*B**2*H**2*P*Q)/(A*B**2*H**2*R**2)/48.0
c
       GVO = -((12*PI**2*A22*A**2*B*H**2*Q**2+12*PI**2*A66*B**3*H**2
    &     *P**2)*R**2+12*PI**2*B66*B**3*H**2*P**2*R+3*PI**2*D66*B**3*
```

```
     &   H**2*P**2)/(A*B**2*H**2*R**2)/48.0
c
c        ** corresponding to w **
c
         AW = 0.0
c
         BW = 0.0
c
         CW1 = (-16*PI**4*H22*A**4*Q**4+((-64*PI**4*H66-32*PI**4*H12)*
     &   A**2*B**2*P**2-9*PI**2*A44*A**4*B**2*H**4+72*PI**2*D44*A**4
     &   *B**2*H**2-144*PI**2*F44*A**4*B**2)*Q**2-16*PI**4*H11*B**4*
     &   P**4+((-9*PI**2*A55*A**2*B**4*H**4+72*PI**2*D55*A**2*B**4*H
     &   **2-144*PI**2*F55*A**2*B**4)*P**2))/(A**3*B**3*H**4)/36.0
c
         CW2 = ((32*PI**4*I22*A**4*Q**4+((64*PI**4*I66+32*PI**4*I12)*A
     &   **2*B**2*P**2-24*PI**2*E22*A**4*B**2*H**2)*Q**2-24*PI**2*
     &   E12*A**2*B**4*H**2*P**2)*R-16*PI**4*J22*A**4*Q**4+(24*PI**2
     &   *F22*A**4*B**2*H**2-16*PI**4*J66*A**2*B**2*P**2)*Q**2-9*A22
     &   *A**4*B**4*H**4)/(A**3*B**3*H**4*R**2)/36.0
c
         CW = CW1+CW2
c
         EW = (((24*PI**3*E66+12*PI**3*E12)*A**3*B**2*H**2*P*Q**2+12*
     &   PI**3*E11*A*B**4*H**2*P**3)*R**2+((-24*PI**3*F66-12*PI**3*
     &   F12)*A**3*B**2*H**2*P*Q**2+9*PI*A12*A**3*B**4*H**4*P)*R+6*
     &   PI**3*G66*A**3*B**2*H**2*P*Q**2)/(A**3*B**3*H**4*R**2)/36.0
c
         GW = ((12*PI**3*E22*A**4*B*H**2*Q**3+(24*PI**3*E66+12*PI**3*
     &   E12)*A**2*B**3*H**2*P**2*Q)*R**2+(9*PI*A22*A**4*B**3*H**4*Q
     &   -12*PI**3*F22*A**4*B*H**2*Q**3)*R-6*PI**3*G66*A**2*B**3*H**
     &   2*P**2*Q)/(A**3*B**3*H**4*R**2)/36.0
c
c        ** corresponding to Ψx **
c
         AJX = -(((9*PI**2*D66*A**2*H**4-24*PI**2*F66*A**2*H**2+16*PI
     &   **2*H66*A**2)*Q**2+(9*PI**2*D11*B**2*H**4-24*PI**2*F11*B**2
     &   *H**2+16*PI**2*H11*B**2)*P**2+9*A55*A**2*B**2*H**4-72*D55*A
     &   **2*B**2*H**2+144*F55*A**2*B**2)*R**2+(-18*PI**2*E66*A**2*H
     &   **4+48*PI**2*G66*A**2*H**2-32*PI**2*I66*A**2)*Q**2*R+(9*PI
     &   **2*F66*A**2*H**4-24*PI**2*H66*A**2*H**2+16*PI**2*J66*A**2)
     &   *Q**2)/(A*B*H**4*R**2)/36.0
c
         BJX = -(((9*PI**2*D26*A**2*H**4-24*PI**2*F26*A**2*H**2+16*PI
```

```
     &    **2*H26*A**2)*Q**2+(9*PI**2*D16*B**2*H**4-24*PI**2*F16*B**2
     &    *H**2+16*PI**2*H16*B**2)*P**2+9*A45*A**2*B**2*H**4-72*D45*A
     &    **2*B**2*H**2+144*F45*A**2*B**2)*R**2+(-18*PI**2*E26*A**2*H
     &    **4+48*PI**2*G26*A**2*H**2-32*PI**2*I26*A**2)*Q**2*R+(9*PI
     &    **2*F26*A**2*H**4-24*PI**2*H26*A**2*H**2+16*PI**2*J26*A**2)
     &    *Q**2)/(A*B*H**4*R**2)/36.0
c
      CJX = 0.0
c
      EJX = 0.0
c
      GJX = 0.0
c
c       ** corresponding to Ψy **
c
      AJY = -(((9*PI**2*D26*A**2*H**4-24*PI**2*F26*A**2*H**2+16*PI
     &    **2*H26*A**2)*Q**2+(9*PI**2*D16*B**2*H**4-24*PI**2*F16*B**2
     &    *H**2+16*PI**2*H16*B**2)*P**2+9*A45*A**2*B**2*H**4-72*D45*A
     &    **2*B**2*H**2+144*F45*A**2*B**2)*R**2+(-18*PI**2*E26*A**2*H
     &    **4+48*PI**2*G26*A**2*H**2-32*PI**2*I26*A**2)*Q**2*R+(9*PI
     &    **2*F26*A**2*H**4-24*PI**2*H26*A**2*H**2+16*PI**2*J26*A**2)
     &    *Q**2)/(A*B*H**4*R**2)/36.0
c
      BJY = -(((9*PI**2*D22*A**2*H**4-24*PI**2*F22*A**2*H**2+16*PI
     &    **2*H22*A**2)*Q**2+(9*PI**2*D66*B**2*H**4-24*PI**2*F66*B**2
     &    *H**2+16*PI**2*H66*B**2)*P**2+9*A44*A**2*B**2*H**4-72*D44*A
     &    **2*B**2*H**2+144*F44*A**2*B**2)*R**2+(-18*PI**2*E22*A**2*H
     &    **4+48*PI**2*G22*A**2*H**2-32*PI**2*I22*A**2)*Q**2*R+(9*PI
     &    **2*F22*A**2*H**4-24*PI**2*H22*A**2*H**2+16*PI**2*J22*A**2)
     &    *Q**2)/(A*B*H**4*R**2)/36.0
c
c
      CJY = 0.0
c
      EJY = 0.0
      GJY = 0.0
c
      Else If (M.EQ.P.AND.MOD(N+Q,2).NE.0) Then
c
c
*****************************************************************
*   The following equations correspond to the Galerkin       *
*            Equations for Case 2                  *
```

```
*******************************************************************
c
c         ** corresponding to u_o **
c
      AUO = ((12*PI*B16*H**2-16*PI*E16)*N*P*Q*R+(12*PI*F16-9*PI*D16
   &    *H**2)*N*P*Q)/((6*H**2*Q**2-6*H**2*N**2)*R)
c
      BUO = (((6*PI*B66+6*PI*B12)*H**2-8*PI*E66-8*PI*E12)*N*P*Q*R+(
   &    (-3*PI*D66-6*PI*D12)*H**2+4*PI*F66+8*PI*F12)*N*P*Q)/(((6*H**
   &    2*Q**2-6*H**2*N**2)*R)
c
      CUO = 0.0
c
      EUO = 0.0
c
      GUO = 0.0
c
c
c         ** corresponding to v_o **
c
      AVO = (((6*PI**2*B26*A**2*H**2-8*PI**2*E26*A**2)*N*Q**2+(6*PI
   &    **2*B16*B**2*H**2-8*PI**2*E16*B**2)*N*P**2)*R+(8*PI**2*F26*
   &    A**2-6*PI**2*D26*A**2*H**2)*N*Q**2+(3*PI**2*D16*B**2*H**2-4
   &    *PI**2*F16*B**2)*N*P**2)/((6*PI*A*B*H**2*Q**2-6*PI*A*B*H**2
   &    *N**2)*R)
c
      BVO = (((6*PI**2*B22*A**2*H**2-8*PI**2*E22*A**2)*N*Q**2+(6*PI
   &    **2*B66*B**2*H**2-8*PI**2*E66*B**2)*N*P**2)*R+(8*PI**2*F22*
   &    A**2-6*PI**2*D22*A**2*H**2)*N*Q**2+(3*PI**2*D66*B**2*H**2-4
   &    *PI**2*F66*B**2)*N*P**2)/((6*PI*A*B*H**2*Q**2-6*PI*A*B*H**2
   &    *N**2)*R)
c
      CVO = 0.0
c
      EVO = 0.0
c
      GVO = 0.0
c
c         ** corresponding to w **
c
      AW1 = -(((36*PI**2*F16*B**2*H**2-48*PI**2*H16*B**2)*N*P**2+(
   &    12*PI**2*F26*A**2*H**2-16*PI**2*H26*A**2)*N**3+(-9*A45*A**2
   &    *B**2*H**4+72*D45*A**2*B**2*H**2-144*F45*A**2*B**2)*N)*Q*R
```

```
     &    **2+((32*PI**2*I16*B**2-24*PI**2*G16*B**2*H**2)*N*P**2+(32*
     &    PI**2*I26*A**2-24*PI**2*G26*A**2*H**2)*N**3+(9*B26*A**2*B**
     &    2*H**4-12*E26*A**2*B**2*H**2)*N)*Q*R+((12*PI**2*H26*A**2*H
     &    **2-16*PI**2*J26*A**2)*N**3+(12*F26*A**2*B**2*H**2-9*D26*A
     &    **2*B**2*H**4)*N)*Q)

      AW = AW1/((9*A*B**2*H**4*Q**2-9*A*B**2*H**4*N**2)*R**2)
c
      BW1 = -(((24*PI**2*F66+12*PI**2*F12)*B**2*H**2+(-32*PI**2*H66
     &    -16*PI**2*H12)*B**2)*N*P**2+(12*PI**2*F22*A**2*H**2-16*PI**
     &    2*H22*A**2)*N**3+(-9*A44*A**2*B**2*H**4+72*D44*A**2*B**2*H
     &    **2-144*F44*A**2*B**2)*N)*Q*R**2-(((-12*PI**2*G66-12*PI**2*
     &    G12)*B**2*H**2+(16*PI**2*I66+16*PI**2*I12)*B**2)*N*P**2+(32
     &    *PI**2*I22*A**2-24*PI**2*G22*A**2*H**2)*N**3+(9*B22*A**2*B
     &    **2*H**4-12*E22*A**2*B**2*H**2)*N)*Q*R-((12*PI**2*H22*A**2*
     &    H**2-16*PI**2*J22*A**2)*N**3+(12*F22*A**2*B**2*H**2-9*D22*A
     &    **2*B**2*H**4)*N)*Q
c
      BW = BW1/((9*A*B**2*H**4*Q**2-9*A*B**2*H**4*N**2)*R**2)
c
      CW = 0.0
c
      EW = 0.0
c
      GW = 0.0
c
c       ** corresponding to Ψx **
c
      AJX = 0.0
c
      BJX = 0.0
      CJX1 = -(((96*PI**2*H16*B**2-72*PI**2*F16*B**2*H**2)*N*P**2+(
     &    32*PI**2*H26*A**2-24*PI**2*F26*A**2*H**2)*N**3+(18*A45*A**2
     &    *B**2*H**4-144*D45*A**2*B**2*H**2+288*F45*A**2*B**2)*N)*Q*R
     &    **2+((48*PI**2*G16*B**2*H**2-64*PI**2*I16*B**2)*N*P**2+(48*
     &    PI**2*G26*A**2*H**2-64*PI**2*I26*A**2)*N**3+(24*E26*A**2*B
     &    **2*H**2-18*B26*A**2*B**2*H**4)*N)*Q*R+((32*PI**2*J26*A**2-
     &    24*PI**2*H26*A**2*H**2)*N**3+(18*D26*A**2*B**2*H**4-24*F26*
     &    A**2*B**2*H**2)*N)*Q)

      CJX = CJX1/((18*A*B**2*H**4*Q**2-18*A*B**2*H**4*N**2)*R**2)
c
      EJX = -((36*PI*B16*A*B**2*H**4-48*PI*E16*A*B**2*H**2)*N*P*Q*R
```

```
     &    **2+(36*PI*F16*A*B**2*H**2-27*PI*D16*A*B**2*H**4)*N*P*Q*R)/
     &    ((18*A*B**2*H**4*Q**2-18*A*B**2*H**4*N**2)*R**2)
c
      GJX = -(((18*PI*B16*B**3*H**4-24*PI*E16*B**3*H**2)*P**2+(18*
     &    PI*B26*A**2*B*H**4-24*PI*E26*A**2*B*H**2)*N**2)*Q*R**2+((9*
     &    PI*D16*B**3*H**4-12*PI*F16*B**3*H**2)*P**2+(24*PI*F26*A**2*
     &    B*H**2-18*PI*D26*A**2*B*H**4)*N**2)*Q*R)/((18*A*B**2*H**4*Q
     &    **2-18*A*B**2*H**4*N**2)*R**2)
c
c      ** corresponding to Ψy **
c
      AJY = 0.0
c
      BJY = 0.0
c
      CJY1 = -(((-48*PI**2*F66-24*PI**2*F12)*B**2*H**2+(64*PI**2*
     &    H66+32*PI**2*H12)*B**2)*N*P**2+(32*PI**2*H22*A**2-24*PI**2*
     &    F22*A**2*H**2)*N**3+(18*A44*A**2*B**2*H**4-144*D44*A**2*B**
     &    2*H**2+288*F44*A**2*B**2)*N)*Q*R**2-(((24*PI**2*G66+24*PI**
     &    2*G12)*B**2*H**2+(-32*PI**2*I66-32*PI**2*I12)*B**2)*N*P**2+
     &    (48*PI**2*G22*A**2*H**2-64*PI**2*I22*A**2)*N**3+(24*E22*A**
     &    2*B**2*H**2-18*B22*A**2*B**2*H**4)*N)*Q*R-((32*PI**2*J22*A
     &    **2-24*PI**2*H22*A**2*H**2)*N**3+(18*D22*A**2*B**2*H**4-24*
     &    F22*A**2*B**2*H**2)*N)*Q
c
      CJY = CJY1/((18*A*B**2*H**4*Q**2-18*A*B**2*H**4*N**2)*R**2)
c
      EJY = -(((18*PI*B66+18*PI*B12)*A*B**2*H**4+(-24*PI*E66-24*PI*
     &    E12)*A*B**2*H**2)*N*P*Q*R**2+((-9*PI*D66-18*PI*D12)*A*B**2*
     &    H**4+(12*PI*F66+24*PI*F12)*A*B**2*H**2)*N*P*Q*R)/((18*A*B**
     &    2*H**4*Q**2-18*A*B**2*H**4*N**2)*R**2)
c
      GJY = -(((18*PI*B66*B**3*H**4-24*PI*E66*B**3*H**2)*P**2+(18*
     &    PI*B22*A**2*B*H**4-24*PI*E22*A**2*B*H**2)*N**2)*Q*R**2+((9*
     &    PI*D66*B**3*H**4-12*PI*F66*B**3*H**2)*P**2+(24*PI*F22*A**2*
     &    B*H**2-18*PI*D22*A**2*B*H**4)*N**2)*Q*R)/((18*A*B**2*H**4*Q
     &    **2-18*A*B**2*H**4*N**2)*R**2)
c
      Else If (MOD(M+P,2).NE.0.AND.N.EQ.Q) Then
c
*****************************************************************
*   The following equations correspond to the Galerkin         *
*             Equations for Case 3                   *
```

```
*****************************************************************c
c
c       ** corresponding to u_o **
c
      AUO = (((6*PI*B66*A**2*H**2-8*PI*E66*A**2)*M*Q**2+(8*PI*E11*B
     &  **2-6*PI*B11*B**2*H**2)*M*P**2+(12*PI*B11*B**2*H**2-16*PI*
     &  E11*B**2)*M**3)*R**2+(12*PI*F66*A**2-9*PI*D66*A**2*H**2)*M*
     &  Q**2*R+(3*PI*E66*A**2*H**2-4*PI*G66*A**2)*M*Q**2)/((6*A*B*H
     &  **2*P**2-6*A*B*H**2*M**2)*R**2)
c
      BUO = (((6*PI*B26*A**2*H**2-8*PI*E26*A**2)*M*Q**2+(8*PI*E16*B
     &  **2-6*PI*B16*B**2*H**2)*M*P**2+(12*PI*B16*B**2*H**2-16*PI*
     &  E16*B**2)*M**3)*R**2+(12*PI*F26*A**2-9*PI*D26*A**2*H**2)*M*
     &  Q**2*R+(3*PI*E26*A**2*H**2-4*PI*G26*A**2)*M*Q**2)/((6*A*B*H
     &  **2*P**2-6*A*B*H**2*M**2)*R**2)
c
      CUO = 0.0
c
      EUO = 0.0
c
      GUO = 0.0
c
c       ** corresponding to v_o **
c
      AVO = (((6*PI*B66+6*PI*B12)*H**2-8*PI*E66-8*PI*E12)*M*P*Q*R**
     &  2+(4*PI*F66-3*PI*D66*H**2)*M*P*Q*R+(4*PI*G66-3*PI*E66*H**2)
     &  *M*P*Q)/((6*H**2*P**2-6*H**2*M**2)*R**2)
c
      BVO = ((12*PI*B26*H**2-16*PI*E26)*M*P*Q*R**2+(4*PI*F26-3*PI*
     &  D26*H**2)*M*P*Q*R+(4*PI*G26-3*PI*E26*H**2)*M*P*Q)/((6*H**2*
     &  P**2-6*H**2*M**2)*R**2)
c
      CVO = 0.0
c
      EVO = 0.0
c
      GVO = 0.0
c
c       ** corresponding to w **
c
      AW = -((((24*PI**2*F66+12*PI**2*F12)*A**2*H**2+(-32*PI**2*H66
     &  -16*PI**2*H12)*A**2)*M*P*Q**2+((12*PI**2*F11*B**2*H**2-16*
     &  PI**2*H11*B**2)*M**3+(-9*A55*A**2*B**2*H**4+72*D55*A**2*B**
```

```
 &    2*H**2-144*F55*A**2*B**2)*M)*P)*R**2+(((-36*PI**2*G66-12*PI
 &    **2*G12)*A**2*H**2+(48*PI**2*I66+16*PI**2*I12)*A**2)*M*P*Q
 &    **2+(9*B12*A**2*B**2*H**4-12*E12*A**2*B**2*H**2)*M*P)*R+(12
 &    *PI**2*H66*A**2*H**2-16*PI**2*J66*A**2)*M*P*Q**2)/((9*A**2*
 &    B*H**4*P**2-9*A**2*B*H**4*M**2)*R**2)
c
      BW = -(((36*PI**2*F26*A**2*H**2-48*PI**2*H26*A**2)*M*P*Q**2+(
 &    (12*PI**2*F16*B**2*H**2-16*PI**2*H16*B**2)*M**3+(-9*A45*A**
 &    2*B**2*H**4+72*D45*A**2*B**2*H**2-144*F45*A**2*B**2)*M)*P)*
 &    R**2+((64*PI**2*I26*A**2-48*PI**2*G26*A**2*H**2)*M*P*Q**2+(
 &    9*B26*A**2*B**2*H**4-12*E26*A**2*B**2*H**2)*M*P)*R+(12*PI**
 &    2*H26*A**2*H**2-16*PI**2*J26*A**2)*M*P*Q**2)/((9*A**2*B*H**
 &    4*P**2-9*A**2*B*H**4*M**2)*R**2)
c
      CW = 0.0
c
      GW = 0.0
c
      EW = 0.0
c
c       ** corresponding to Ψx **
c
      AJX = 0.0
c
      BJX = 0.0
c
      CJX = -((((-48*PI**2*F66-24*PI**2*F12)*A**2*H**2+(64*PI**2*
 &    H66+32*PI**2*H12)*A**2)*M*P*Q**2+((32*PI**2*H11*B**2-24*PI
 &    **2*F11*B**2*H**2)*M**3+(18*A55*A**2*B**2*H**4-144*D55*A**2
 &    *B**2*H**2+288*F55*A**2*B**2)*M)*P)*R**2+(((72*PI**2*G66+24
 &    *PI**2*G12)*A**2*H**2+(-96*PI**2*I66-32*PI**2*I12)*A**2)*M*
 &    P*Q**2+(24*E12*A**2*B**2*H**2-18*B12*A**2*B**2*H**4)*M*P)*R
 &    +(32*PI**2*J66*A**2-24*PI**2*H66*A**2*H**2)*M*P*Q**2)/((18*
 &    A**2*B*H**4*P**2-18*A**2*B*H**4*M**2)*R**2)
c
      EJX = -(((18*PI*B66*A**3*H**4-24*PI*E66*A**3*H**2)*P*Q**2+(18
 &    *PI*B11*A*B**2*H**4-24*PI*E11*A*B**2*H**2)*M**2*P)*R**2+(36
 &    *PI*F66*A**3*H**2-27*PI*D66*A**3*H**4)*P*Q**2*R+(9*PI*E66*A
 &    **3*H**4-12*PI*G66*A**3*H**2)*P*Q**2)/((18*A**2*B*H**4*P**2
 &    -18*A**2*B*H**4*M**2)*R**2)
c
      GJX = -(((18*PI*B66+18*PI*B12)*A**2*B*H**4+(-24*PI*E66-24*PI*
 &    E12)*A**2*B*H**2)*M*P*Q*R**2+(12*PI*F66*A**2*B*H**2-9*PI*
```

```
    &     D66*A**2*B*H**4)*M*P*Q*R+(12*PI*G66*A**2*B*H**2-9*PI*E66*A
    &     **2*B*H**4)*M*P*Q)/((18*A**2*B*H**4*P**2-18*A**2*B*H**4*M**
    &     2)*R**2)
c
c        ** corresponding to Ψy **
c
      AJY = 0.0
c
      BJY = 0.0
c
      CJY = -(((96*PI**2*H26*A**2-72*PI**2*F26*A**2*H**2)*M*P*Q**2+
    &     ((32*PI**2*H16*B**2-24*PI**2*F16*B**2*H**2)*M**3+(18*A45*A
    &     **2*B**2*H**4-144*D45*A**2*B**2*H**2+288*F45*A**2*B**2)*M)*
    &     P)*R**2+((96*PI**2*G26*A**2*H**2-128*PI**2*I26*A**2)*M*P*Q
    &     **2+(24*E26*A**2*B**2*H**2-18*B26*A**2*B**2*H**4)*M*P)*R+(
    &     32*PI**2*J26*A**2-24*PI**2*H26*A**2*H**2)*M*P*Q**2)/((18*A
    &     **2*B*H**4*P**2-18*A**2*B*H**4*M**2)*R**2)
c
      EJY = -((((18*PI*B26*A**3*H**4-24*PI*E26*A**3*H**2)*P*Q**2+(18
    &     *PI*B16*A*B**2*H**4-24*PI*E16*A*B**2*H**2)*M**2*P)*R**2+(36
    &     *PI*F26*A**3*H**2-27*PI*D26*A**3*H**4)*P*Q**2*R+(9*PI*E26*A
    &     **3*H**4-12*PI*G26*A**3*H**2)*P*Q**2)/((18*A**2*B*H**4*P**2
    &     -18*A**2*B*H**4*M**2)*R**2)

c
      GJY = -((36*PI*B26*A**2*B*H**4-48*PI*E26*A**2*B*H**2)*M*P*Q*R
    &     **2+(12*PI*F26*A**2*B*H**2-9*PI*D26*A**2*B*H**4)*M*P*Q*R+(
    &     12*PI*G26*A**2*B*H**2-9*PI*E26*A**2*B*H**4)*M*P*Q)/((18*A**
    &     2*B*H**4*P**2-18*A**2*B*H**4*M**2)*R**2)
c
c
      Else If (MOD(M+P,2).NE.0.AND.MOD(N+Q,2).NE.0) Then
c
******************************************************************
*    The following equations correspond to the Galerkin          *
*              Equations for Case 4                    *
******************************************************************
c
c        ** corresponding to uo **
c
      AUO = 0.0
      BUO = 0.0
      CUO = ((-32*PI**2*E16*B**2*M*N*P**2-16*PI**2*E26*A**2*M*N**3-
```

```
     &    16*PI**2*E16*B**2*M**3*N)*Q*R**2+(16*PI**2*F16*B**2*M*N*P**
     &    2+24*PI**2*F26*A**2*M*N**3+(8*PI**2*F16*B**2*M**3-12*A26*A
     &    **2*B**2*H**2*M)*N)*Q*R+(6*B26*A**2*B**2*H**2*M*N-8*PI**2*
     &    G26*A**2*M*N**3)*Q)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H
     &    **2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*
     &    M**2*N**2)*R**2)
c
      EUO = ((12*PI*A16*A*B**2*H**2*N*P**2+12*PI*A16*A*B**2*H**2*M
     &    **2*N)*Q*R**2+(-6*PI*B16*A*B**2*H**2*N*P**2-6*PI*B16*A*B**2
     &    *H**2*M**2*N)*Q*R)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**
     &    2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M
     &    **2*N**2)*R**2)
c
      GUO = ((12*PI*A16*B**3*H**2*M*P**2+12*PI*A26*A**2*B*H**2*M*N
     &    **2)*Q*R**2+(6*PI*B16*B**3*H**2*M*P**2-6*PI*B26*A**2*B*H**2
     &    *M*N**2)*Q*R)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**2*M**
     &    2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M**2*N
     &    **2)*R**2)
c
c      ** corresponding to v_o **
c
      AVO = 0.0
      BVO = 0.0
c
      CVO = (((-16*PI**2*E26*A**2*M*N**3-16*PI**2*E16*B**2*M**3*N)*
     &    P-32*PI**2*E26*A**2*M*N*P*Q**2)*R**2+(16*PI**2*F26*A**2*M*N
     &    *P*Q**2+(8*PI**2*F26*A**2*M*N**3+(-8*PI**2*F16*B**2*M**3-12
     &    *A26*A**2*B**2*H**2*M)*N)*P)*R+(8*PI**2*G26*A**2*M*N**3-6*
     &    B26*A**2*B**2*H**2*M*N)*P)/(((3*PI*A**2*B*H**2*P**2-3*PI*A
     &    **2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*
     &    B*H**2*M**2*N**2)*R**2)
c
      EVO = ((12*PI*A26*A**3*H**2*N*P*Q**2+12*PI*A16*A*B**2*H**2*M
     &    **2*N*P)*R**2+(6*PI*B16*A*B**2*H**2*M**2*N*P-6*PI*B26*A**3*
     &    H**2*N*P*Q**2)*R)/(((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H**2
     &    *M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**
     &    2*N**2)*R**2)
c
      GVO = ((12*PI*A26*A**2*B*H**2*M*P*Q**2+12*PI*A26*A**2*B*H**2*
     &    M*N**2*P)*R**2+(6*PI*B26*A**2*B*H**2*M*P*Q**2+6*PI*B26*A**2
     &    *B*H**2*M*N**2*P)*R)/(((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H
     &    **2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*
     &    M**2*N**2)*R**2)
```

```
c
c       ** corresponding to w **
c
      AW = 0.0
c
      BW = 0.0
c
      CW = -((((-72*PI*A45*A**2*B**2*H**4+576*PI*D45*A**2*B**2*H**2
     &  -1152*PI*F45*A**2*B**2)*M-256*PI**3*H16*B**2*M**3)*N-256*PI
     &  **3*H26*A**2*M*N**3)*P*Q*R**2+(384*PI**3*I26*A**2*M*N**3+(
     &  128*PI**3*I16*B**2*M**3-192*PI*E26*A**2*B**2*H**2*M)*N)*P*Q
     &  *R+(96*PI*F26*A**2*B**2*H**2*M*N-128*PI**3*J26*A**2*M*N**3)
     &  *P*Q)/(((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)
     &  *Q**2-9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**
     &  2*N**2)*R**2)
c
      EW = -((48*PI**2*E26*A**3*H**2*N**3+144*PI**2*E16*A*B**2*H**2
     &  *M**2*N)*P*Q*R**2+((36*A26*A**3*B**2*H**4-72*PI**2*F16*A*B
     &  **2*H**2*M**2)*N-72*PI**2*F26*A**3*H**2*N**3)*P*Q*R+(24*PI
     &  **2*G26*A**3*H**2*N**3-18*B26*A**3*B**2*H**4*N)*P*Q)/(((9*
     &  PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*
     &  A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)*R**
     &  2)
c
      GW = -((144*PI**2*E26*A**2*B*H**2*M*N**2+48*PI**2*E16*B**3*H
     &  **2*M**3)*P*Q*R**2+(-72*PI**2*F26*A**2*B*H**2*M*N**2+24*PI
     &  **2*F16*B**3*H**2*M**3+36*A26*A**2*B**3*H**4*M)*P*Q*R+(18*
     &  B26*A**2*B**3*H**4*M-24*PI**2*G26*A**2*B*H**2*M*N**2)*P*Q)/
     &  (((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-
     &  9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2
     &  )*R**2)
c
c       ** corresponding to Ψx **
c
      AJX = ((72*D16*H**4-192*F16*H**2+128*H16)*M*N*P*Q*R+(-72*E16*
     &  H**4+192*G16*H**2-128*I16)*M*N*P*Q)/(((9*H**4*P**2-9*H**4*M
     &  **2)*Q**2-9*H**4*N**2*P**2+9*H**4*M**2*N**2)*R)
c
      BJX = (((36*D66+36*D12)*H**4+(-96*F66-96*F12)*H**2+64*H66+64*
     &  H12)*M*N*P*Q*R+((-36*E66-36*E12)*H**4+(96*G66+96*G12)*H**2-
     &  64*I66-64*I12)*M*N*P*Q)/(((9*H**4*P**2-9*H**4*M**2)*Q**2-9*
     &  H**4*N**2*P**2+9*H**4*M**2*N**2)*R)
```

```
c
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
c
c       ** corresponding to Ψy **
c
      AJY = (((36*D66+36*D12)*H**4+(-96*F66-96*F12)*H**2+64*H66+64*
   &    H12)*M*N*P*Q*R+((-36*E66-36*E12)*H**4+(96*G66+96*G12)*H**2-
   &    64*I66-64*I12)*M*N*P*Q)/(((9*H**4*P**2-9*H**4*M**2)*Q**2-9*
   &    H**4*N**2*P**2+9*H**4*M**2*N**2)*R)
c
      BJY = ((72*D26*H**4-192*F26*H**2+128*H26)*M*N*P*Q*R+(-72*E26*
   &    H**4+192*G26*H**2-128*I26)*M*N*P*Q)/(((9*H**4*P**2-9*H**4*M
   &    **2)*Q**2-9*H**4*N**2*P**2+9*H**4*M**2*N**2)*R)
c
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
     Else
c
      AUO = 0.0
      BUO = 0.0
      CUO = 0.0
      EUO = 0.0
      GUO = 0.0
      AVO = 0.0
      BVO = 0.0
      CVO = 0.0
      EVO = 0.0
      GVO = 0.0
      AW = 0.0
      BW = 0.0
      CW = 0.0
      EW = 0.0
      GW = 0.0
      AJX = 0.0
      BJX = 0.0
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
      AJY = 0.0
      BJY = 0.0
```

```
      CJY = 0.0
      EJY = 0.0
      GJY = 0.0
      End If
c
c  STORE THESE TERMS IN THE STIFFNESS MATRIX
c-----------------------------------------------
      STIFF(I,J) = AUO
      STIFF(I,J+MMAX*NMAX) = BUO
      STIFF(I,J+2*MMAX*NMAX) = CUO
      STIFF(I,J+3*MMAX*NMAX) = EUO
      STIFF(I,J+4*MMAX*NMAX) = GUO
      STIFF(I+MMAX*NMAX,J) = AVO
      STIFF(I+MMAX*NMAX,J+MMAX*NMAX) = BVO
      STIFF(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVO
      STIFF(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVO
      STIFF(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVO
      STIFF(I+2*MMAX*NMAX,J) = AW
      STIFF(I+2*MMAX*NMAX,J+MMAX*NMAX) = BW
      STIFF(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CW
      STIFF(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EW
      STIFF(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GW
      STIFF(I+3*MMAX*NMAX,J) = AJX
      STIFF(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJX
      STIFF(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJX
      STIFF(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJX
      STIFF(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJX
      STIFF(I+4*MMAX*NMAX,J) = AJY
      STIFF(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJY
      STIFF(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJY
      STIFF(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJY
      STIFF(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJY
c------------------------------------------------------------
c  COMPUTE MASS MATRIX ELEMENTS
c------------------------------------------------------------
c  FIRST CALCULATE THE MASS MOMENTS OF INERTIA.
      I2BARPR = RHO*H**3/(15.0*R)
      I3BARPR = RHO*H**3/(60.0*R)
      I5BAR = RHO*H**3*4.0/315.0
      I7 = RHO*H**7/448.0
      I1 = RHO*H
      I4BAR = RHO*H**3*17.0/315.0
      AUOMASS = 0.0
```

```
      BUOMASS = 0.0
      CUOMASS = 0.0
      EUOMASS = 0.0
      GUOMASS = 0.0
      AVOMASS = 0.0
      EVOMASS = 0.0
      GVOMASS = 0.0
      EWMASS = 0.0
      GWMASS = 0.0
      BJXMASS = 0.0
      EJXMASS = 0.0
      GJXMASS = 0.0
      AJYMASS = 0.0
      EJYMASS = 0.0
      GJYMASS = 0.0
c
c
      If (NBUCVIB.EQ.1) Then
c   VIBRATIONS PROBLEM - WE ARE LOOKING FOR THE NATURAL
c    FREQUENCIES
c
       If (M.EQ.P.AND.N.EQ.Q) Then
        BVOMASS = 0.0
        CVOMASS = PI*A*I3BARPR*Q/4.0
        AWMASS = 0.0
        BWMASS = 0.0
        CWMASS = -(16*PI**2*A**2*I7*Q**2+16*PI**2*B**2*I7*P**2+9*A**
     &     2*B**2*H**4*I1)/(A*B*H**4)/36.0
        AJXMASS = -A*B*I4BAR/4.0
        CJXMASS = 0.0
        BJYMASS = -A*B*I4BAR/4.0
        CJYMASS = 0.0
c
       Else If (M.EQ.P.AND.MOD(N+Q,2).NE.0) Then
        BVOMASS = A*B*I2BARPR*N/(PI*Q**2-N**2*PI)
        CVOMASS = 0.0
        AWMASS = 0.0
        BWMASS = -A*I5BAR*N*Q/(Q**2-N**2)
        CWMASS = 0.0
        AJXMASS = 0.0
        CJXMASS = 0.0
        BJYMASS = 0.0
        CJYMASS = A*I5BAR*N*Q/(Q**2-N**2)
```

```
c
      Else If (MOD(M+P,2).NE.0.AND.N.EQ.Q) Then
       BVOMASS = 0.0
       CVOMASS = 0.0
       AWMASS = -B*I5BAR*M*P/(P**2-M**2)
       BWMASS = 0.0
       CWMASS = 0.0
       AJXMASS = 0.0
       CJXMASS = B*I5BAR*M*P/(P**2-M**2)
       BJYMASS = 0.0
       CJYMASS = 0.0
c
      Else

       BVOMASS = 0.0
       CVOMASS = 0.0
       AWMASS = 0.0
       BWMASS = 0.0
       CWMASS = 0.0
       AJXMASS = 0.0
       CJXMASS = 0.0
       BJYMASS = 0.0
       CJYMASS = 0.0
c
      End If
c
     Else
c
c  BUCKLING PROBLEM - WE ARE LOOKING FOR THE CRITICAL BUCKLING
c   LOADS
c
      BVOMASS = 0.0
      CVOMASS = 0.0
      AWMASS = 0.0
      BWMASS = 0.0
      AJXMASS = 0.0
      CJXMASS = 0.0
      BJYMASS = 0.0
      CJYMASS = 0.0
c
      If (M.EQ.P.AND.N.EQ.Q) Then
       CWMASS = B*P**2*PI**2/A/4.0
      Else
```

```
         CWMASS = 0.0
c
        End If
c
       End If
c
c
c-----------------------------------------------------------
c   STORE THESE TERMS IN THE MASS MATRIX
c-----------------------------------------------------------
       MASS(I,J) = AUOMASS
       MASS(I,J+MMAX*NMAX) = BUOMASS
       MASS(I,J+2*MMAX*NMAX) = CUOMASS
       MASS(I,J+3*MMAX*NMAX) = EUOMASS
       MASS(I,J+4*MMAX*NMAX) = GUOMASS
       MASS(I+MMAX*NMAX,J) = AVOMASS
       MASS(I+MMAX*NMAX,J+MMAX*NMAX) = BVOMASS
       MASS(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVOMASS
       MASS(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVOMASS
       MASS(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVOMASS
       MASS(I+2*MMAX*NMAX,J) = AWMASS
       MASS(I+2*MMAX*NMAX,J+MMAX*NMAX) = BWMASS
       MASS(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CWMASS
       MASS(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EWMASS
       MASS(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GWMASS
       MASS(I+3*MMAX*NMAX,J) = AJXMASS
       MASS(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJXMASS
       MASS(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJXMASS
       MASS(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJXMASS
       MASS(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJXMASS
       MASS(I+4*MMAX*NMAX,J) = AJYMASS
       MASS(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJYMASS
       MASS(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJYMASS
       MASS(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJYMASS
       MASS(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJYMASS
c-----------------------------------------------------------
       J = J+1
  20   Continue
      I = I+1
      J = 1
  10 Continue
c-----------------------------------------------------------
c   CALL THE IMSL LIBRARY SUBROUTINE. USE THE MASS AND STIFFNESS
```

```
c    MATRICES AS INPUT AND FIND THE EIGENVALUES AND EIGENVECTORS.
     Call DGVCRG(MSIZE,STIFF,MSIZE,MASS,MSIZE,ALPHA,BETA,EVEC,MSIZE)
     Do 40 I = 1,MSIZE
      If (BETA(I).NE.0.0) Then
       EVAL(I) = ALPHA(I)/BETA(I)
      Else
       EVAL(I) = (1.0D+30,0.0D+00)
      End If
   40 Continue
     If (NBUCVIB.EQ.1) Then
c    PRINT OUT THE FIRST 10 MODES FOR THE VIBRATION PROBLEM
c
      Do 50 I = 1,10
       REVAL = DREAL(EVAL(I))
       AGEVAL = DIMAG(EVAL(I))
       If (ABS(AGEVAL).GT.1.0D-15) Then
        Write (2,115) I
       Else If (REVAL.GT.1.0D+28) Then
        Write (2,125) I
       Else If (REVAL.LT.0.0) Then
        Write (2,120) I
       Else
        OMEGA = SQRT(REVAL)
        Write (2,130) I,REVAL,OMEGA
       End If
   50  Continue
c
     Else
c
c    PRINT OUT THE CRITICAL BUCKLING LOAD. THE CRITICAL
c    BUCKLING LOAD IS THE EIGENVALUE WITH THE SMALLEST ABSOLUTE
c    VALUE .
c
      Do 55 I = 2,MSIZE
       If (ABS(DIMAG(EVAL(I-1))).GT.1.0D-15) Then
        Go To 55
       End If
       If (ABS(DREAL(EVAL(I))).GT.ABS(DREAL(EVAL(I-1))).AND.
    &   ABS(DREAL(EVAL(I-1))).LT.1.0D+28) Then
        Write (2,220) DREAL(EVAL(I-1))
       End If
   55  Continue
c
```

```
      End If
c
c     PRINT OUT THE 1ST MODE OF THE DEFLECTION, W(X,Y), ALONG THE
c     MIDLINES OF THE PANEL: X = A/2 AND Y = B/2
c
c     PRINT OUT THE W EIGENVECTOR, CMN
c
      II = 1
      Write (2,500)
      Write (2,510)
      MNWMIN = 1+2*MMAX*NMAX
      MNWMAX = 3*MMAX*NMAX
      Do 400 I = MNWMIN,MNWMAX
       REVEC(II) = DREAL(EVEC(I,1))
       AGEVEC = DIMAG(EVEC(I,1))
       If (ABS(AGEVEC).GT.1.0D-15) Then
        Write (2,520) I,II,REVEC(II)
       Else
        Write (2,530) I,II,REVEC(II)
       End If
       II = II+1
  400 Continue
c
c     DETERMINE W(X=A/2,Y)
c
      ASTEP = A/50.0
      BSTEP = B/50.0
      XCOORD = A/2.0
      YCOORD = 0.0
      Write (2,540)
      Write (2,542)
  801 WMODE = 0.0
      JJJ = 1
      Do 470 M = 1,MMAX
       Do 472 N = 1,NMAX
        WMODE = WMODE+REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOORD/B)
        JJJ = JJJ+1
  472  Continue
  470 Continue
      Write (2,550) YCOORD,WMODE
      YCOORD = YCOORD+BSTEP
      If (YCOORD.GT.B) Then
       Go To 800
```

```
      Else
       Go To 801
      End If
c
 800 YCOORD = B/2.0
c
c     DETERMINE W(X, Y=B/2)
c
      XCOORD = 0.0
      Write (2,560)
      Write (2,570)
 810 WMODE = 0.0
      JJJ = 1
      Do 480 M = 1,MMAX
       Do 482 N = 1,NMAX
        WMODE = WMODE+REVEC(JJJ)*SIN(M*PI*XCOORD/A)*SIN(N*PI*YCOORD/B)
        JJJ = JJJ+1
 482  Continue
 480 Continue
      Write (2,550) XCOORD,WMODE
      XCOORD = XCOORD+ASTEP
      If (XCOORD.GT.A) Then
       Go To 850
      Else
       Go To 810
      End If
c-----------------------------------------------------------------
 115 Format (/,8X,I3,11X,'EIGENVALUE IS COMPLEX')
 120 Format (/,9X,I3,11X,'EIGENVALUE IS NEGATIVE')
 125 Format (/,9X,I3,11X,'EIGENVALUE IS INFINITE')
 130 Format (/,9X,I3,10X,D20.13,12X,D20.13)
 200 Format (/,8X,I3,10X,D20.13)
 220 Format (//,5X,'CRITICAL BUCKLING LOAD = ',1X,D20.13)
 500 Format (//,5X,'W EIGENVECTOR, CMN, FOR 1ST MODE')
 510 Format (//,5X,'M, N',10X,'CMN')
 520 Format (/,5X,I4,2X,I4,12X,D20.13,3X,'COMPLEX')
 530 Format (/,5X,I4,2X,I4,12X,D20.13)
 540 Format (//,5X,'DEFLECTION, W(X=A/2,Y)')
 542 Format (//,5X,'Y(IN.)',10X,'W(A/2,Y)(IN.)')
 550 Format (/,5X,F6.2,11X,E15.8)
 560 Format (//,5X,'DEFLECTION, W(X,Y=B/2)')
 570 Format (//,5X,'X(IN.)',10X,'W(X, B/2)(IN.)')
 850 Return
```

The following is the GALERK subroutine for the simple-clamped boundary

```
c------------------------------------------------------------
   Subroutine GALERK(PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
  & D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,
  & F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,
  & B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,
  & I11,I12,I22,I16,I26,I66,NBUCVIB,MMAX,MSIZE,RHO,STIFF,MASS,BETA,
  & ALPHA,EVAL,EVEC,MSIZESQ,REVEC)
c------------------------------------------------------------
c  THIS SUBROUTINE GENERATES THE GALERKIN EQUATIONS AND FORMS
c  THE MASS AND STIFFNESS MATRICES. THEN IT CALLS DGVCRG, AN
c  IMSL SUBROUTINE WHICH SOLVES THE EIGENVALUE PROBLEM:
c
c     [STIFF]{X} = (OMEGA^2 OR N1BAR)[MASS]{X}
c------------------------------------------------------------
   Double Precision PI,R,H,A,B,A11,A12,A22,A16,A26,A66,A44,A45,A55,
  & D11,D12,D22,D16,D26,D66,D44,D45,D55,F11,F12,F22,F16,F26,F66,F44,
  & F45,F55,H11,H12,H22,H16,H26,H66,J11,J12,J22,J16,J26,J66,B11,B12,
  & B22,B16,B26,B66,E11,E12,E22,E16,E26,E66,G11,G12,G22,G16,G26,G66,
  & I11,I12,I22,I16,I26,I66,STIFF(MSIZE,MSIZE),MASS(MSIZE,MSIZE),
  & AUO,BUO,CUO,EUO,GUO,AVO,BVO,CVO,EVO,GVO,AW,BW,CW,EW,GW,AJX,BJX,
  & CJX,EJX,GJX,AJY,BJY,CJY,EJY,GJY,AUOMASS,BUOMASS,CUOMASS,EUOMASS,
  & GUOMASS,AVOMASS,BVOMASS
   Double Precision CVOMASS,EVOMASS,GVOMASS,AWMASS,BWMASS,CWMASS,
  & EWMASS,GWMASS,AJXMASS,BJXMASS,CJXMASS,EJXMASS,GJXMASS,AJYMASS,
  & BJYMASS,CJYMASS,EJYMASS,GJYMASS,RHO,I2BARPR,I3BARPR,I5BAR,I7,I1,
  & I4BAR
   Integer P,Q,M,N,MMAX,NMAX
c  THESE VARIABLES NEEDED FOR THE IMSL EIGENVALUE SOLVER .
   Double Precision BETA(MSIZE),REVAL,OMEGA,AGEVAL,AGEVEC,
  & REVEC(MSIZESQ)
   Double Complex ALPHA(MSIZE),EVAL(MSIZE),EVEC(MSIZE,MSIZE)
c------------------------------------------------------------
c  NUMBER OF TERMS IN THE ADMISSIBLE FUNCTIONS
   NMAX = MMAX
c  GENERATE GALERKIN EQUATIONS
   I = 1
   J = 1
   Do 10 P = 1,MMAX
    Do 10 Q = 1,NMAX
     Do 20 M = 1,MMAX
      Do 20 N = 1,NMAX
```

```
c-------------------------------------------------------------
c   COMPUTE STIFFNESS MATRIX ELEMENTS
c-------------------------------------------------------------
*****************************************************************
*    The following equations correspond to the Galerkin        *
*              Equations for Case I             *
*****************************************************************
c
      If (M.EQ.P.AND.N.EQ.Q) Then
c
c          ** corresponding to uo
c
        AUO = 0.0
        BUO = -(((12*PI**2*B66+12*PI**2*B12)*A**2*B*H**2+(-16*PI**2*
     &    E66-16*PI**2*E12)*A**2*B)*P*Q*R**2+((-6*PI**2*D66-12*PI**2*
     &    D12)*A**2*B*H**2+(8*PI**2*F66+16*PI**2*F12)*A**2*B)*P*Q*R)/
     &    (A**2*B*H**2*R**2)/48.0
c
        CUO = -(((-32*PI**3*E66-16*PI**3*E12)*A**2*P*Q**2-16*PI**3*
     &    E11*B**2*P**3)*R**2+((32*PI**3*F66+16*PI**3*F12)*A**2*P*Q**
     &    2-12*PI*A12*A**2*B**2*H**2*P)*R-8*PI**3*G66*A**2*P*Q**2)/(A
     &    **2*B*H**2*R**2)/48.0
c
        EUO = -((12*PI**2*A66*A**3*H**2*Q**2+12*PI**2*A11*A*B**2*H**2
     &    *P**2)*R**2-12*PI**2*B66*A**3*H**2*Q**2*R+3*PI**2*D66*A**3*
     &    H**2*Q**2)/(A**2*B*H**2*R**2)/48.0
c
        GUO = -((12*PI**2*A66+12*PI**2*A12)*A**2*B*H**2*P*Q*R**2-3*PI
     &    **2*D66*A**2*B*H**2*P*Q)/(A**2*B*H**2*R**2)/48.0
c
c          ** corresponding to vo **
c
        AVO = 0.0
c
        BVO = -(((12*PI**2*B22*A**2*B*H**2-16*PI**2*E22*A**2*B)*Q**2+
     &    (12*PI**2*B66*B**3*H**2-16*PI**2*E66*B**3)*P**2)*R**2+((16*
     &    PI**2*F22*A**2*B-12*PI**2*D22*A**2*B*H**2)*Q**2+(6*PI**2*
     &    D66*B**3*H**2-8*PI**2*F66*B**3)*P**2)*R)/(A*B**2*H**2*R**2)
     &    /48.0
c
        CVO = -((((-32*PI**3*E66-16*PI**3*E12)*B**2*P**2)*Q-16*PI**3*
     &    E22*A**2*Q**3)*R**2+(16*PI**3*F22*A**2*Q**3-12*PI*A22*A**2*
     &    B**2*H**2*Q)*R+8*PI**3*G66*B**2*P**2*Q)/(A*B**2*H**2*R**2)/
```

```
     &   48.0
c
      EVO = -((12*PI**2*A66+12*PI**2*A12)*A*B**2*H**2*P*Q*R**2-3*PI
     &   **2*D66*A*B**2*H**2*P*Q)/(A*B**2*H**2*R**2)/48.0
c
      GVO = -((12*PI**2*A22*A**2*B*H**2*Q**2+12*PI**2*A66*B**3*H**2
     &   *P**2)*R**2+12*PI**2*B66*B**3*H**2*P**2*R+3*PI**2*D66*B**3*
     &   H**2*P**2)/(A*B**2*H**2*R**2)/48.0
c
c       ** corresponding to w **
c
      AW = 0.0
c
      BW1 = ((12*PI**3*F22*A**4*B*H**2-16*PI**3*H22*A**4*B)*Q**3+((
     &   (24*PI**3*F66+12*PI**3*F12)*A**2*B**3*H**2+(-32*PI**3*H66-
     &   16*PI**3*H12)*A**2*B**3)*P**2-9*PI*A44*A**4*B**3*H**4+72*PI
     &   *D44*A**4*B**3*H**2-144*PI*F44*A**4*B**3)*Q)/(A**3*B**3*H**
     &   4)/36.0
c
      BW2 = (((32*PI**3*I22*A**4*B-24*PI**3*G22*A**4*B*H**2)*Q**3+(
     &   ((-12*PI**3*G66-12*PI**3*G12)*A**2*B**3*H**2+(16*PI**3*I66+
     &   16*PI**3*I12)*A**2*B**3)*P**2+9*PI*B22*A**4*B**3*H**4-12*PI
     &   *E22*A**4*B**3*H**2)*Q)*R+(12*PI**3*H22*A**4*B*H**2-16*PI**
     &   3*J22*A**4*B)*Q**3+(12*PI*F22*A**4*B**3*H**2-9*PI*D22*A**4*
     &   B**3*H**4)*Q)/(A**3*B**3*H**4*R**2)/36.0
c
      BW = BW1+BW2
c
      CW1 = (-16*PI**4*H22*A**4*Q**4+((-64*PI**4*H66-32*PI**4*H12)*
     &   A**2*B**2*P**2-9*PI**2*A44*A**4*B**2*H**4+72*PI**2*D44*A**4
     &   *B**2*H**2-144*PI**2*F44*A**4*B**2)*Q**2-16*PI**4*H11*B**4*
     &   P**4+(-9*PI**2*A55*A**2*B**4*H**4+72*PI**2*D55*A**2*B**4*H
     &   **2-144*PI**2*F55*A**2*B**4)*P**2)/(A**3*B**3*H**4)/36.0
c
      CW2 = ((32*PI**4*I22*A**4*Q**4+((64*PI**4*I66+32*PI**4*I12)*A
     &   **2*B**2*P**2-24*PI**2*E22*A**4*B**2*H**2)*Q**2-24*PI**2*
     &   E12*A**2*B**4*H**2*P**2)*R-16*PI**4*J22*A**4*Q**4+(24*PI**2
     &   *F22*A**4*B**2*H**2-16*PI**4*J66*A**2*B**2*P**2)*Q**2-9*A22
     &   *A**4*B**4*H**4)/(A**3*B**3*H**4*R**2)/36.0
c
      CW = CW1+CW2
c
      EW = (((24*PI**3*E66+12*PI**3*E12)*A**3*B**2*H**2*P*Q**2+12*
```

```
   &    PI**3*E11*A*B**4*H**2*P**3)*R**2+((-24*PI**3*F66-12*PI**3*
   &    F12)*A**3*B**2*H**2*P*Q**2+9*PI*A12*A**3*B**4*H**4*P)*R+6*
   &    PI**3*G66*A**3*B**2*H**2*P*Q**2)/(A**3*B**3*H**4*R**2)/36.0
c
      GW = ((12*PI**3*E22*A**4*B*H**2*Q**3+(24*PI**3*E66+12*PI**3*
   &    E12)*A**2*B**3*H**2*P**2*Q)*R**2+(9*PI*A22*A**4*B**3*H**4*Q
   &    -12*PI**3*F22*A**4*B*H**2*Q**3)*R-6*PI**3*G66*A**2*B**3*H**
   &    2*P**2*Q)/(A**3*B**3*H**4*R**2)/36.0
c
c        ** corresponding to Ψx **
c
      AJX = -(((9*PI**2*D66*A**2*H**4-24*PI**2*F66*A**2*H**2+16*PI
   &    **2*H66*A**2)*Q**2+(9*PI**2*D11*B**2*H**4-24*PI**2*F11*B**2
   &    *H**2+16*PI**2*H11*B**2)*P**2+9*A55*A**2*B**2*H**4-72*D55*A
   &    **2*B**2*H**2+144*F55*A**2*B**2)*R**2+(-18*PI**2*E66*A**2*H
   &    **4+48*PI**2*G66*A**2*H**2-32*PI**2*I66*A**2)*Q**2*R+(9*PI
   &    **2*F66*A**2*H**4-24*PI**2*H66*A**2*H**2+16*PI**2*J66*A**2)
   &    *Q**2)/(A*B*H**4*R**2)/36.0
c
      BJX = 0.0
c
      CJX = 0.0
c
      EJX = 0.0
c
      GJX = 0.0
c
      AJY = 0.0
c
c        ** corresponding to Ψy **
c
      BJY = -(((18*PI**2*D22*A**2*B*H**4-48*PI**2*F22*A**2*B*H**2+
   &    32*PI**2*H22*A**2*B)*Q**2+(18*PI**2*D66*B**3*H**4-48*PI**2*
   &    F66*B**3*H**2+32*PI**2*H66*B**3)*P**2+18*A44*A**2*B**3*H**4
   &    -144*D44*A**2*B**3*H**2+288*F44*A**2*B**3)*R**2+(-36*PI**2*
   &    E22*A**2*B*H**4+96*PI**2*G22*A**2*B*H**2-64*PI**2*I22*A**2*
   &    B)*Q**2*R+(18*PI**2*F22*A**2*B*H**4-48*PI**2*H22*A**2*B*H**
   &    2+32*PI**2*J22*A**2*B)*Q**2)/(A*B**2*H**4*R**2)/72.0
c
c
      CJY1 = -((32*PI**3*H22*A**2-24*PI**3*F22*A**2*H**2)*Q**3+(((
   &    -48*PI**3*F66-24*PI**3*F12)*B**2*H**2+(64*PI**3*H66+32*PI**
   &    3*H12)*B**2)*P**2+18*PI*A44*A**2*B**2*H**4-144*PI*D44*A**2*
```

```
     &    B**2*H**2+288*PI*F44*A**2*B**2)*Q)/(A*B**2*H**4)/72.0
c
      CJY2 = -(((48*PI**3*G22*A**2*H**2-64*PI**3*I22*A**2)*Q**3+(((
     &    24*PI**3*G66+24*PI**3*G12)*B**2*H**2+(-32*PI**3*I66-32*PI**
     &    3*I12)*B**2)*P**2-18*PI*B22*A**2*B**2*H**4+24*PI*E22*A**2*B
     &    **2*H**2)*Q)*R+(32*PI**3*J22*A**2-24*PI**3*H22*A**2*H**2)*Q
     &    **3+(18*PI*D22*A**2*B**2*H**4-24*PI*F22*A**2*B**2*H**2)*Q)/
     &    (A*B**2*H**4*R**2)/72.0
c
      CJY = CJY1+CJY2
c
      EJY = -(((18*PI**2*B66+18*PI**2*B12)*A*B**2*H**4+(-24*PI**2*
     &    E66-24*PI**2*E12)*A*B**2*H**2)*P*Q*R**2+((-9*PI**2*D66-18*
     &    PI**2*D12)*A*B**2*H**4+(12*PI**2*F66+24*PI**2*F12)*A*B**2*H
     &    **2)*P*Q*R)/(A*B**2*H**4*R**2)/72.0
c
      GJY = -(((18*PI**2*B22*A**2*B*H**4-24*PI**2*E22*A**2*B*H**2)*
     &    Q**2+(18*PI**2*B66*B**3*H**4-24*PI**2*E66*B**3*H**2)*P**2)*
     &    R**2+((24*PI**2*F22*A**2*B*H**2-18*PI**2*D22*A**2*B*H**4)*Q
     &    **2+(9*PI**2*D66*B**3*H**4-12*PI**2*F66*B**3*H**2)*P**2)*R)
     &    /(A*B**2*H**4*R**2)/72.0
c
     Else If (M.EQ.P.AND.MOD(N+Q,2).NE.0) Then
c
*******************************************************************
*    The following equations correspond to the Galerkin      *
*              Equations for Case 2                  *
*******************************************************************
c
c        ** corresponding to uo **
c
      AUO = ((12*PI*B16*H**2-16*PI*E16)*N*P*Q*R+(12*PI*F16-9*PI*D16
     &    *H**2)*N*P*Q)/(((6*H**2*Q**2-6*H**2*N**2)*R)
c
      BUO = 0.0
c
      CUO = 0.0
c
      EUO = 0.0
c
      GUO = 0.0
c
c        ** corresponding to vo **
```

```
c
      AVO = (((6*PI*B26*A**2*H**2-8*PI*E26*A**2)*N*Q**2+(6*PI*B16*B
     &  **2*H**2-8*PI*E16*B**2)*N*P**2)*R+(8*PI*F26*A**2-6*PI*D26*A
     &  **2*H**2)*N*Q**2+(3*PI*D16*B**2*H**2-4*PI*F16*B**2)*N*P**2)
     &  /((6*A*B*H**2*Q**2-6*A*B*H**2*N**2)*R)
c
      BVO = 0.0
c
      CVO = 0.0
c
      EVO = 0.0
c
      GVO = 0.0
c
c       ** corresponding to w **
c
      AW = -(((36*PI**2*F16*B**2*H**2-48*PI**2*H16*B**2)*N*P**2+(12
     &  *PI**2*F26*A**2*H**2-16*PI**2*H26*A**2)*N**3+(-9*A45*A**2*B
     &  **2*H**4+72*D45*A**2*B**2*H**2-144*F45*A**2*B**2)*N)*Q*R**2
     &  +((32*PI**2*I16*B**2-24*PI**2*G16*B**2*H**2)*N*P**2+(32*PI
     &  **2*I26*A**2-24*PI**2*G26*A**2*H**2)*N**3+(9*B26*A**2*B**2*
     &  H**4-12*E26*A**2*B**2*H**2)*N)*Q*R+((12*PI**2*H26*A**2*H**2
     &  -16*PI**2*J26*A**2)*N**3+(12*F26*A**2*B**2*H**2-9*D26*A**2*
     &  B**2*H**4)*N)*Q)/((9*A*B**2*H**4*Q**2-9*A*B**2*H**4*N**2)*R
     &  **2)
c
      BW = 0.0
c
      CW = 0.0
c
      EW = 0.0
c
      GW = 0.0
c
c       ** corresponding to Ψx **
c
      AJX = 0.0
c
      BJX = -(((18*PI**2*D16*B**3*H**4-48*PI**2*F16*B**3*H**2+32*PI
     &  **2*H16*B**3)*P**2+(18*PI**2*D26*A**2*B*H**4-48*PI**2*F26*A
     &  **2*B*H**2+32*PI**2*H26*A**2*B)*N**2+18*A45*A**2*B**3*H**4-
     &  144*D45*A**2*B**3*H**2+288*F45*A**2*B**3)*Q*R**2+(-36*PI**2
     &  *E26*A**2*B*H**4+96*PI**2*G26*A**2*B*H**2-64*PI**2*I26*A**2
```

```fortran
     &    *B)*N**2*Q*R+(18*PI**2*F26*A**2*B*H**4-48*PI**2*H26*A**2*B*
     &    H**2+32*PI**2*J26*A**2*B)*N**2*Q)/((18*PI*A*B**2*H**4*Q**2-
     &    18*PI*A*B**2*H**4*N**2)*R**2)
c
      CJX = -(((96*PI**3*H16*B**2-72*PI**3*F16*B**2*H**2)*N*P**2+(
     &    32*PI**3*H26*A**2-24*PI**3*F26*A**2*H**2)*N**3+(18*PI*A45*A
     &    **2*B**2*H**4-144*PI*D45*A**2*B**2*H**2+288*PI*F45*A**2*B**
     &    2)*N)*Q*R**2+((48*PI**3*G16*B**2*H**2-64*PI**3*I16*B**2)*N*
     &    P**2+(48*PI**3*G26*A**2*H**2-64*PI**3*I26*A**2)*N**3+(24*PI
     &    *E26*A**2*B**2*H**2-18*PI*B26*A**2*B**2*H**4)*N)*Q*R+((32*
     &    PI**3*J26*A**2-24*PI**3*H26*A**2*H**2)*N**3+(18*PI*D26*A**2
     &    *B**2*H**4-24*PI*F26*A**2*B**2*H**2)*N)*Q)/((18*PI*A*B**2*H
     &    **4*Q**2-18*PI*A*B**2*H**4*N**2)*R**2)
c
      EJX = -((36*PI**2*B16*A*B**2*H**4-48*PI**2*E16*A*B**2*H**2)*N
     &    *P*Q*R**2+(36*PI**2*F16*A*B**2*H**2-27*PI**2*D16*A*B**2*H**
     &    4)*N*P*Q*R)/((18*PI*A*B**2*H**4*Q**2-18*PI*A*B**2*H**4*N**2
     &    )*R**2)
c
      GJX = -(((18*PI**2*B16*B**3*H**4-24*PI**2*E16*B**3*H**2)*P**2
     &    +(18*PI**2*B26*A**2*B*H**4-24*PI**2*E26*A**2*B*H**2)*N**2)*
     &    Q*R**2+((9*PI**2*D16*B**3*H**4-12*PI**2*F16*B**3*H**2)*P**2
     &    +(24*PI**2*F26*A**2*B*H**2-18*PI**2*D26*A**2*B*H**4)*N**2)*
     &    Q*R)/((18*PI*A*B**2*H**4*Q**2-18*PI*A*B**2*H**4*N**2)*R**2)
c
c       ** corresponding to Ψy **
c
      AJY1 = ((9*PI**2*D26*A**2*H**4-36*PI**2*F26*A**2*H**2+32*PI**
     &    2*H26*A**2)*N*Q**2+(9*PI**2*D16*B**2*H**4-24*PI**2*F16*B**2
     &    *H**2+16*PI**2*H16*B**2)*N*P**2+(12*PI**2*F26*A**2*H**2-16*
     &    PI**2*H26*A**2)*N**3+(9*A45*A**2*B**2*H**4-72*D45*A**2*B**2
     &    *H**2+144*F45*A**2*B**2)*N)/(9*PI*A*B*H**4*Q**2-9*PI*A*B*H
     &    **4*N**2)
c
      AJY2 = (((-18*PI**2*E26*A**2*H**4+72*PI**2*G26*A**2*H**2-64*
     &    PI**2*I26*A**2)*N*Q**2+(32*PI**2*I26*A**2-24*PI**2*G26*A**2
     &    *H**2)*N**3)*R+(9*PI**2*F26*A**2*H**4-36*PI**2*H26*A**2*H**
     &    2+32*PI**2*J26*A**2)*N*Q**2+(12*PI**2*H26*A**2*H**2-16*PI**
     &    2*J26*A**2)*N**3)/((9*PI*A*B*H**4*Q**2-9*PI*A*B*H**4*N**2)*
     &    R**2)
c
      AJY = AJY1+AJY2
c
```

```
      BJY = 0.0
c
      CJY = 0.0
c
      EJY = 0.0
c
      GJY = 0.0
c
      Else If (MOD(M+P,2).NE.0.AND.N.EQ.Q) Then
c
*****************************************************************
*    The following equations correspond to the Galerkin     *
*              Equations for Case 3               *
*****************************************************************
c         ** corresponding to uo **
c
      AUO = (((6*PI*B66*A**2*H**2-8*PI*E66*A**2)*M*Q**2+(6*PI*B11*B
     &    **2*H**2-8*PI*E11*B**2)*M*P**2)*R**2+(12*PI*F66*A**2-9*PI*
     &    D66*A**2*H**2)*M*Q**2*R+(3*PI*E66*A**2*H**2-4*PI*G66*A**2)*
     &    M*Q**2)/((6*A*B*H**2*P**2-6*A*B*H**2*M**2)*R**2)
c
      BUO = 0.0
c
      CUO = 0.0
c
      EUO = 0.0
c
      GUO = 0.0
c
c         ** corresponding to vo **
c
      AVO = (((6*PI*B66+6*PI*B12)*H**2-8*PI*E66-8*PI*E12)*M*P*Q*R**
     &    2+(4*PI*F66-3*PI*D66*H**2)*M*P*Q*R+(4*PI*G66-3*PI*E66*H**2)
     &    *M*P*Q)/((6*H**2*P**2-6*H**2*M**2)*R**2)
c
      BVO = 0.0
c
      CVO = 0.0
c
      EVO = 0.0
c
      GVO = 0.0
c
```

```
c        ** coresponding to w **
c
      AW = -((((24*PI**2*F66+12*PI**2*F12)*A**2*H**2+(-32*PI**2*H66
     &   -16*PI**2*H12)*A**2)*M*P*Q**2+((12*PI**2*F11*B**2*H**2-16*
     &   PI**2*H11*B**2)*M**3+(-9*A55*A**2*B**2*H**4+72*D55*A**2*B**
     &   2*H**2-144*F55*A**2*B**2)*M)*P)*R**2+(((-36*PI**2*G66-12*PI
     &   **2*G12)*A**2*H**2+(48*PI**2*I66+16*PI**2*I12)*A**2)*M*P*Q
     &   **2+(9*B12*A**2*B**2*H**4-12*E12*A**2*B**2*H**2)*M*P)*R+(12
     &   *PI**2*H66*A**2*H**2-16*PI**2*J66*A**2)*M*P*Q**2)/((9*A**2*
     &   B*H**4*P**2-9*A**2*B*H**4*M**2)*R**2)
c
      BW = 0.0
c
      CW = 0.0
c
      GW = 0.0
c
      EW = 0.0
c
c        ** corresponding to Ψx **
c
      AJX = 0.0
c
      BJX = -(((18*PI*D66+18*PI*D12)*A**2*B*H**4+(-48*PI*F66-48*PI*
     &   F12)*A**2*B*H**2+(32*PI*H66+32*PI*H12)*A**2*B)*M*P*Q*R**2+(
     &   (-18*PI*E66-18*PI*E12)*A**2*B*H**4+(48*PI*G66+48*PI*G12)*A
     &   **2*B*H**2+(-32*PI*I66-32*PI*I12)*A**2*B)*M*P*Q*R)/((18*A**
     &   2*B*H**4*P**2-18*A**2*B*H**4*M**2)*R**2)
c
      CJX = -((((-48*PI**2*F66-24*PI**2*F12)*A**2*H**2+(64*PI**2*
     &   H66+32*PI**2*H12)*A**2)*M*P*Q**2+((32*PI**2*H11*B**2-24*PI
     &   **2*F11*B**2*H**2)*M**3+(18*A55*A**2*B**2*H**4-144*D55*A**2
     &   *B**2*H**2+288*F55*A**2*B**2)*M)*P)*R**2+(((72*PI**2*G66+24
     &   *PI**2*G12)*A**2*H**2+(-96*PI**2*I66-32*PI**2*I12)*A**2)*M*
     &   P*Q**2+(24*E12*A**2*B**2*H**2-18*B12*A**2*B**2*H**4)*M*P)*R
     &   +(32*PI**2*J66*A**2-24*PI**2*H66*A**2*H**2)*M*P*Q**2)/((18*
     &   A**2*B*H**4*P**2-18*A**2*B*H**4*M**2)*R**2)
c
      EJX = -(((18*PI*B66*A**3*H**4-24*PI*E66*A**3*H**2)*P*Q**2+(18
     &   *PI*B11*A*B**2*H**4-24*PI*E11*A*B**2*H**2)*M**2*P)*R**2+(36
     &   *PI*F66*A**3*H**2-27*PI*D66*A**3*H**4)*P*Q**2*R+(9*PI*E66*A
     &   **3*H**4-12*PI*G66*A**3*H**2)*P*Q**2)/((18*A**2*B*H**4*P**2
     &   -18*A**2*B*H**4*M**2)*R**2)
```

```
c
      GJX = -(((18*PI*B66+18*PI*B12)*A**2*B*H**4+(-24*PI*E66-24*PI*
     &  E12)*A**2*B*H**2)*M*P*Q*R**2+(12*PI*F66*A**2*B*H**2-9*PI*
     &  D66*A**2*B*H**4)*M*P*Q*R+(12*PI*G66*A**2*B*H**2-9*PI*E66*A
     &  **2*B*H**4)*M*P*Q)/((18*A**2*B*H**4*P**2-18*A**2*B*H**4*M**
     &  2)*R**2)
c
c      ** corresponding to Ψy **
c
      AJY = (((9*PI*D66+9*PI*D12)*H**4+(-24*PI*F66-24*PI*F12)*H**2+
     &  16*PI*H66+16*PI*H12)*M*P*Q*R+((-9*PI*E66-9*PI*E12)*H**4+(24
     &  *PI*G66+24*PI*G12)*H**2-16*PI*I66-16*PI*I12)*M*P*Q)/((9*H**
     &  4*P**2-9*H**4*M**2)*R)
      BJY = 0.0
c
      CJY = 0.0
c
      EJY = 0.0
c
      GJY = 0.0
c
c
      Else If (MOD(M+P,2).NE.0.AND.MOD(N+Q,2).NE.0) Then
c
*****************************************************************
*   The following equations correspond to the Galerkin        *
*             Equations for Case 4                    *
*****************************************************************
c
c      ** corresponding to uo **
c
      AUO = 0.0
c
      BUO = (((12*PI*B16*B**3*H**2-16*PI*E16*B**3)*M*P**2+(12*PI*
     &  B26*A**2*B*H**2-16*PI*E26*A**2*B)*M*N**2)*Q*R**2+(24*PI*F26
     &  *A**2*B-18*PI*D26*A**2*B*H**2)*M*N**2*Q*R+(6*PI*E26*A**2*B*
     &  H**2-8*PI*G26*A**2*B)*M*N**2*Q)/(((3*PI*A*B**2*H**2*P**2-3*
     &  PI*A*B**2*H**2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A
     &  *B**2*H**2*M**2*N**2)*R**2)
c
      CUO = ((-32*PI**2*E16*B**2*M*N*P**2-16*PI**2*E26*A**2*M*N**3-
     &  16*PI**2*E16*B**2*M**3*N)*Q*R**2+(16*PI**2*F16*B**2*M*N*P**
     &  2+24*PI**2*F26*A**2*M*N**3+(8*PI**2*F16*B**2*M**3-12*A26*A
```

```
&     **2*B**2*H**2*M)*N)*Q*R+(6*B26*A**2*B**2*H**2*M*N-8*PI**2*
&     G26*A**2*M*N**3)*Q)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H
&     **2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*
&     M**2*N**2)*R**2)
c
      EUO = ((12*PI*A16*A*B**2*H**2*N*P**2+12*PI*A16*A*B**2*H**2*M
&     **2*N)*Q*R**2+(-6*PI*B16*A*B**2*H**2*N*P**2-6*PI*B16*A*B**2
&     *H**2*M**2*N)*Q*R)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**
&     2*M**2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M
&     **2*N**2)*R**2)
c
      GUO = ((12*PI*A16*B**3*H**2*M*P**2+12*PI*A26*A**2*B*H**2*M*N
&     **2)*Q*R**2+(6*PI*B16*B**3*H**2*M*P**2-6*PI*B26*A**2*B*H**2
&     *M*N**2)*Q*R)/(((3*PI*A*B**2*H**2*P**2-3*PI*A*B**2*H**2*M**
&     2)*Q**2-3*PI*A*B**2*H**2*N**2*P**2+3*PI*A*B**2*H**2*M**2*N
&     **2)*R**2)
c
c      ** corresponding to v_o **
c
      AVO = 0.0
c
      BVO = (((12*PI*B26*A**2*B*H**2-16*PI*E26*A**2*B)*M*P*Q**2+(12
&     *PI*B26*A**2*B*H**2-16*PI*E26*A**2*B)*M*N**2*P)*R**2+(8*PI*
&     F26*A**2*B-6*PI*D26*A**2*B*H**2)*M*N**2*P*R+(8*PI*G26*A**2*
&     B-6*PI*E26*A**2*B*H**2)*M*N**2*P)/(((3*PI*A**2*B*H**2*P**2-
&     3*PI*A**2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI
&     *A**2*B*H**2*M**2*N**2)*R**2)
c
      CVO = (((-16*PI**2*E26*A**2*M*N**3-16*PI**2*E16*B**2*M**3*N)*
&     P-32*PI**2*E26*A**2*M*N*P*Q**2)*R**2+(16*PI**2*F26*A**2*M*N
&     *P*Q**2+(8*PI**2*F26*A**2*M*N**3+(-8*PI**2*F16*B**2*M**3-12
&     *A26*A**2*B**2*H**2*M)*N)*P)*R+(8*PI**2*G26*A**2*M*N**3-6*
&     B26*A**2*B**2*H**2*M*N)*P)/(((3*PI*A**2*B*H**2*P**2-3*PI*A
&     **2*B*H**2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*
&     B*H**2*M**2*N**2)*R**2)
c
      EVO = ((12*PI*A26*A**3*H**2*N*P*Q**2+12*PI*A16*A*B**2*H**2*M
&     **2*N*P)*R**2+(6*PI*B16*A*B**2*H**2*M**2*N*P-6*PI*B26*A**3*
&     H**2*N*P*Q**2)*R)/(((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H**2
&     *M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*M**
&     2*N**2)*R**2)
c
      GVO = ((12*PI*A26*A**2*B*H**2*M*P*Q**2+12*PI*A26*A**2*B*H**2*
```

```
     &    M*N**2*P)*R**2+(6*PI*B26*A**2*B*H**2*M*P*Q**2+6*PI*B26*A**2
     &    *B*H**2*M*N**2*P)*R)/(((3*PI*A**2*B*H**2*P**2-3*PI*A**2*B*H
     &    **2*M**2)*Q**2-3*PI*A**2*B*H**2*N**2*P**2+3*PI*A**2*B*H**2*
     &    M**2*N**2)*R**2)
c
c       ** corresponding to w **
c
      AW = 0.0
c
      BW = -(((144*PI**2*F26*A**2*B*H**2-192*PI**2*H26*A**2*B)*M*N
     &    **2+(48*PI**2*F16*B**3*H**2-64*PI**2*H16*B**3)*M**3+(
     &    -36*A45*A**2*B**3*H**4+288*D45*A**2*B**3*H**2-576*F45*A**2*
     &    B**3)*M)*P*Q*R**2+((256*PI**2*I26*A**2*B-192*PI**2*G26*A**2
     &    *B*H**2)*M*N**2+(36*B26*A**2*B**3*H**4-48*E26*A**2*B**3*H**
     &    2)*M)*P*Q*R+(48*PI**2*H26*A**2*B*H**2-64*PI**2*J26*A**2*B)*
     &    M*N**2*P*Q)/(((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4
     &    *M**2)*Q**2-9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H
     &    **4*M**2*N**2)*R**2)
c
      CW = -((((-72*PI*A45*A**2*B**2*H**4+576*PI*D45*A**2*B**2*H**2
     &    -1152*PI*F45*A**2*B**2)*M-256*PI**3*H16*B**2*M**3)*N-256*PI
     &    **3*H26*A**2*M*N**3)*P*Q*R**2+(384*PI**3*I26*A**2*M*N**3+(
     &    128*PI**3*I16*B**2*M**3-192*PI*E26*A**2*B**2*H**2*M)*N)*P*Q
     &    *R+(96*PI*F26*A**2*B**2*H**2*M*N-128*PI**3*J26*A**2*M*N**3)
     &    *P*Q)/(((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)
     &    *Q**2-9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**
     &    2*N**2)*R**2)
c
      EW = -((48*PI**2*E26*A**3*H**2*N**3+144*PI**2*E16*A*B**2*H**2
     &    *M**2*N)*P*Q*R**2+((36*A26*A**3*B**2*H**4-72*PI**2*F16*A*B
     &    **2*H**2*M**2)*N-72*PI**2*F26*A**3*H**2*N**3)*P*Q*R+(24*PI
     &    **2*G26*A**3*H**2*N**3-18*B26*A**3*B**2*H**4*N)*P*Q)/(((9*
     &    PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-9*PI*
     &    A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2)*R**
     &    2)
c
      GW = -((144*PI**2*E26*A**2*B*H**2*M*N**2+48*PI**2*E16*B**3*H
     &    **2*M**3)*P*Q*R**2+(-72*PI**2*F26*A**2*B*H**2*M*N**2+24*PI
     &    **2*F16*B**3*H**2*M**3+36*A26*A**2*B**3*H**4*M)*P*Q*R+(18*
     &    B26*A**2*B**3*H**4*M-24*PI**2*G26*A**2*B*H**2*M*N**2)*P*Q)/
     &    (((9*PI*A**2*B**2*H**4*P**2-9*PI*A**2*B**2*H**4*M**2)*Q**2-
     &    9*PI*A**2*B**2*H**4*N**2*P**2+9*PI*A**2*B**2*H**4*M**2*N**2
     &    )*R**2)
```

```
c
c        ** corresponding to Ψx **
c
      AJX = ((72*D16*H**4-192*F16*H**2+128*H16)*M*N*P*Q*R+(-72*E16*
     &  H**4+192*G16*H**2-128*I16)*M*N*P*Q)/(((9*H**4*P**2-9*H**4*M
     &  **2)*Q**2-9*H**4*N**2*P**2+9*H**4*M**2*N**2)*R)
c
      BJX = 0.0
c
      CJX = 0.0
      EJX = 0.0
      GJX = 0.0
c
c        ** corresponding to Ψy **
c
      AJY = 0.0
c
      BJY = (((36*PI*D26*A**2*B*H**4-144*PI*F26*A**2*B*H**2+128*PI*
     &  H26*A**2*B)*M*P*Q**2+(36*PI*D26*A**2*B*H**4-48*PI*F26*A**2*
     &  B*H**2)*M*N**2*P)*R**2+((-36*PI*E26*A**2*B*H**4+144*PI*G26*
     &  A**2*B*H**2-128*PI*I26*A**2*B)*M*P*Q**2+(48*PI*G26*A**2*B*H
     &  **2-36*PI*E26*A**2*B*H**4)*M*N**2*P)*R)/(((9*PI*A**2*B*H**4
     &  *P**2-9*PI*A**2*B*H**4*M**2)*Q**2-9*PI*A**2*B*H**4*N**2*P**
     &  2+9*PI*A**2*B*H**4*M**2*N**2)*R**2)
c
      CJY1 = ((256*PI**2*H26*A**2-96*PI**2*F26*A**2*H**2)*M*N*P*Q**
     &  2+((-48*PI**2*F26*A**2*H**2-64*PI**2*H26*A**2)*M*N**3+((64*
     &  PI**2*H16*B**2-48*PI**2*F16*B**2*H**2)*M**3+(36*A45*A**2*B
     &  **2*H**4-288*D45*A**2*B**2*H**2+576*F45*A**2*B**2)*M)*N)*P)
     &  /((9*PI*A**2*B*H**4*P**2-9*PI*A**2*B*H**4*M**2)*Q**2-9*PI*A
     &  **2*B*H**4*N**2*P**2+9*PI*A**2*B*H**4*M**2*N**2)
c
      CJY2 = (((144*PI**2*G26*A**2*H**2-384*PI**2*I26*A**2)*M*N*P*Q
     &  **2+((48*PI**2*G26*A**2*H**2+128*PI**2*I26*A**2)*M*N**3+(48
     &  *E26*A**2*B**2*H**2-36*B26*A**2*B**2*H**4)*M*N)*P)*R+(128*
     &  PI**2*J26*A**2-48*PI**2*H26*A**2*H**2)*M*N*P*Q**2-64*PI**2*
     &  J26*A**2*M*N**3*P)/(((9*PI*A**2*B*H**4*P**2-9*PI*A**2*B*H**
     &  4*M**2)*Q**2-9*PI*A**2*B*H**4*N**2*P**2+9*PI*A**2*B*H**4*M
     &  **2*N**2)*R**2)
c
      CJY = CJY1+CJY2
c
      EJY = (((36*PI*B26*A**3*H**4-96*PI*E26*A**3*H**2)*N*P*Q**2+(
```

```
     &    48*PI*E26*A**3*H**2*N**3+(36*PI*B16*A*B**2*H**4-48*PI*E16*A
     &    *B**2*H**2)*M**2*N)*P)*R**2+((144*PI*F26*A**3*H**2-54*PI*
     &    D26*A**3*H**4)*N*P*Q**2-72*PI*F26*A**3*H**2*N**3*P)*R+(18*
     &    PI*E26*A**3*H**4-48*PI*G26*A**3*H**2)*N*P*Q**2+24*PI*G26*A
     &    **3*H**2*N**3*P)/(((9*PI*A**2*B*H**4*P**2-9*PI*A**2*B*H**4*
     &    M**2)*Q**2-9*PI*A**2*B*H**4*N**2*P**2+9*PI*A**2*B*H**4*M**2
     &    *N**2)*R**2)
c
        GJY = (((36*PI*B26*A**2*B*H**4-96*PI*E26*A**2*B*H**2)*M*P*Q**
     &    2+36*PI*B26*A**2*B*H**4*M*N**2*P)*R**2+((48*PI*F26*A**2*B*H
     &    **2-18*PI*D26*A**2*B*H**4)*M*P*Q**2-24*PI*F26*A**2*B*H**2*M
     &    *N**2*P)*R+(48*PI*G26*A**2*B*H**2-18*PI*E26*A**2*B*H**4)*M*
     &    P*Q**2-24*PI*G26*A**2*B*H**2*M*N**2*P)/(((9*PI*A**2*B*H**4*
     &    P**2-9*PI*A**2*B*H**4*M**2)*Q**2-9*PI*A**2*B*H**4*N**2*P**2
     &    +9*PI*A**2*B*H**4*M**2*N**2)*R**2)
c
       Else
c
        AUO = 0.0
        BUO = 0.0
        CUO = 0.0
        EUO = 0.0
        GUO = 0.0
        AVO = 0.0
        BVO = 0.0
        CVO = 0.0
        EVO = 0.0
        GVO = 0.0
        AW = 0.0
        BW = 0.0
        CW = 0.0
        EW = 0.0
        GW = 0.0
        AJX = 0.0
        BJX = 0.0
        CJX = 0.0
        EJX = 0.0
        GJX = 0.0
        AJY = 0.0
        BJY = 0.0
        CJY = 0.0
        EJY = 0.0
        GJY = 0.0
```

```
      End If
c
c  STORE THESE TERMS IN THE STIFFNESS MATRIX
c---------------------------------------------
      STIFF(I,J) = AUO
      STIFF(I,J+MMAX*NMAX) = BUO
      STIFF(I,J+2*MMAX*NMAX) = CUO
      STIFF(I,J+3*MMAX*NMAX) = EUO
      STIFF(I,J+4*MMAX*NMAX) = GUO
      STIFF(I+MMAX*NMAX,J) = AVO
      STIFF(I+MMAX*NMAX,J+MMAX*NMAX) = BVO
      STIFF(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVO
      STIFF(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVO
      STIFF(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVO
      STIFF(I+2*MMAX*NMAX,J) = AW
      STIFF(I+2*MMAX*NMAX,J+MMAX*NMAX) = BW
      STIFF(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CW
      STIFF(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EW
      STIFF(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GW
      STIFF(I+3*MMAX*NMAX,J) = AJX
      STIFF(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJX
      STIFF(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJX
      STIFF(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJX
      STIFF(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJX
      STIFF(I+4*MMAX*NMAX,J) = AJY
      STIFF(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJY
      STIFF(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJY
      STIFF(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJY
      STIFF(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJY
c------------------------------------------------------------
C  COMPUTE MASS MATRIX ELEMENTS
c------------------------------------------------------------
c  FIRST CALCULATE THE MASS MOMENTS OF INERTIA.
      I2BARPR = RHO*H**3/(15.0*R)
      I3BARPR = RHO*H**3/(60.0*R)
      I5BAR = RHO*H**3*4.0/315.0
      I7 = RHO*H**7/448.0
      I1 = RHO*H
      I4BAR = RHO*H**3*17.0/315.0
      AUOMASS = 0.0
      BUOMASS = 0.0
      CUOMASS = 0.0
      EUOMASS = 0.0
```

```
      GUOMASS = 0.0
      AVOMASS = 0.0
      EVOMASS = 0.0
      GVOMASS = 0.0
      EWMASS = 0.0
      GWMASS = 0.0
      BJXMASS = 0.0
      EJXMASS = 0.0
      GJXMASS = 0.0
      AJYMASS = 0.0
      EJYMASS = 0.0
      GJYMASS = 0.0
C
      If (NBUCVIB.EQ.1) Then
c   VIBRATIONS PROBLEM - WE ARE LOOKING FOR THE NATURAL
c     FREQUENCIES
c
       If (M.EQ.P.AND.N.EQ.Q) Then
        BVOMASS = -A*B*I2BARPR/4.0
        CVOMASS = PI*A*I3BARPR*Q/4.0
        AWMASS = 0.0
        BWMASS = PI*A*I5BAR*Q/4.0
        CWMASS = -(16*PI**2*A**2*I7*Q**2+16*PI**2*B**2*I7*P**2+9*A**
     &     2*B**2*H**4*I1)/(A*B*H**4)/36.0
        AJXMASS = -A*B*I4BAR/4.0
        CJXMASS = 0.0
        BJYMASS = -A*B*I4BAR/4.0
        CJYMASS = PI*A*I5BAR*Q/4.0
c
       Else If (MOD(M+P,2).NE.0.AND.N.EQ.Q) Then
        BVOMASS = 0.0
        CVOMASS = 0.0
        AWMASS = -B*I5BAR*M*P/(P**2-M**2)
        BWMASS = 0.0
        CWMASS = 0.0
        AJXMASS = 0.0
        CJXMASS = B*I5BAR*M*P/(P**2-M**2)
        BJYMASS = 0.0
        CJYMASS = 0.0
c
       Else

        BVOMASS = 0.0
```

```
          CVOMASS = 0.0
          AWMASS = 0.0
          BWMASS = 0.0
          CWMASS = 0.0
          AJXMASS = 0.0
          CJXMASS = 0.0
          BJYMASS = 0.0
          CJYMASS = 0.0
c
         End If
c
        Else
c
c   BUCKLING PROBLEM - WE ARE LOOKING FOR THE CRITICAL BUCKLING
c    LOADS
c
         BVOMASS = 0.0
         CVOMASS = 0.0
         AWMASS = 0.0
         BWMASS = 0.0
         AJXMASS = 0.0
         CJXMASS = 0.0
         BJYMASS = 0.0
         CJYMASS = 0.0
c
         If (M.EQ.P.AND.N.EQ.Q) Then
          CWMASS = B*P**2*PI**2/A/4.0
         Else
          CWMASS = 0.0
c
         End If
c
        End If
c-----------------------------------------------------------
c   STORE THESE TERMS IN THE MASS MATRIX
c-----------------------------------------------------------
        MASS(I,J) = AUOMASS
        MASS(I,J+MMAX*NMAX) = BUOMASS
        MASS(I,J+2*MMAX*NMAX) = CUOMASS
        MASS(I,J+3*MMAX*NMAX) = EUOMASS
        MASS(I,J+4*MMAX*NMAX) = GUOMASS
        MASS(I+MMAX*NMAX,J) = AVOMASS
        MASS(I+MMAX*NMAX,J+MMAX*NMAX) = BVOMASS
```

```
      MASS(I+MMAX*NMAX,J+2*MMAX*NMAX) = CVOMASS
      MASS(I+MMAX*NMAX,J+3*MMAX*NMAX) = EVOMASS
      MASS(I+MMAX*NMAX,J+4*MMAX*NMAX) = GVOMASS
      MASS(I+2*MMAX*NMAX,J) = AWMASS
      MASS(I+2*MMAX*NMAX,J+MMAX*NMAX) = BWMASS
      MASS(I+2*MMAX*NMAX,J+2*MMAX*NMAX) = CWMASS
      MASS(I+2*MMAX*NMAX,J+3*MMAX*NMAX) = EWMASS
      MASS(I+2*MMAX*NMAX,J+4*MMAX*NMAX) = GWMASS
      MASS(I+3*MMAX*NMAX,J) = AJXMASS
      MASS(I+3*MMAX*NMAX,J+MMAX*NMAX) = BJXMASS
      MASS(I+3*MMAX*NMAX,J+2*MMAX*NMAX) = CJXMASS
      MASS(I+3*MMAX*NMAX,J+3*MMAX*NMAX) = EJXMASS
      MASS(I+3*MMAX*NMAX,J+4*MMAX*NMAX) = GJXMASS
      MASS(I+4*MMAX*NMAX,J) = AJYMASS
      MASS(I+4*MMAX*NMAX,J+MMAX*NMAX) = BJYMASS
      MASS(I+4*MMAX*NMAX,J+2*MMAX*NMAX) = CJYMASS
      MASS(I+4*MMAX*NMAX,J+3*MMAX*NMAX) = EJYMASS
      MASS(I+4*MMAX*NMAX,J+4*MMAX*NMAX) = GJYMASS
c-----------------------------------------------------------------
      J = J+1
      I = I+1
      J = 1
c-----------------------------------------------------------------
c   CALL THE IMSL LIBRARY SUBROUTINE. USE THE MASS AND STIFFNESS
c   MATRICES AS INPUT AND FIND THE EIGENVALUES AND EIGENVECTORS.
      Call DGVCRG(MSIZE,STIFF,MSIZE,MASS,MSIZE,ALPHA,BETA,EVEC,MSIZE
     &   )
      Do 10 I = 1,MSIZE
       If (BETA(I).NE.0.0) Then
        EVAL(I) = ALPHA(I)/BETA(I)
       Else
        EVAL(I) = (1.0D+30,0.0D+00)
       End If
  10   Continue
      If (NBUCVIB.EQ.1) Then
c   PRINT OUT THE FIRST 10 MODES FOR THE VIBRATION PROBLEM
c
       Do 20 I = 1,10
        REVAL = DREAL(EVAL(I))
        AGEVAL = DIMAG(EVAL(I))
        If (ABS(AGEVAL).GT.1.0D-15) Then
         Write (2,5000) I
        Else If (REVAL.GT.1.0D+28) Then
```

```
         Write (2,5200) I
        Else If (REVAL.LT.0.0) Then
         Write (2,5100) I
        Else
         OMEGA = SQRT(REVAL)
         Write (2,5300) I,REVAL,OMEGA
        End If
  20   Continue
c
      Else
c
c   PRINT OUT THE CRITICAL BUCKLING LOAD. THE CRITICAL
c   BUCKLING LOAD IS THE EIGENVALUE WITH THE SMALLEST ABSOLUTE
c   VALUE .
C
       Do 30 I = 2,MSIZE
        If (ABS(DIMAG(EVAL(I-1))).GT.1.0D-15) Then
         Go To 30
        End If
        If (ABS(DREAL(EVAL(I))).GT.ABS(DREAL(EVAL(I-1))).AND.
   &     ABS(DREAL(EVAL(I-1))).LT.1.0D+28) Then
         Write (2,5500) DREAL(EVAL(I-1))
        End If
  30   Continue
c
      End If
c
c   PRINT OUT THE 1ST MODE OF THE DEFLECTION, W(X,Y), ALONG THE
c   MIDLINES OF THE PANEL: X = A/2 AND Y = B/2
c
c   PRINT OUT THE W EIGENVECTOR, CMN
c
      II = 1
      Write (2,5600)
      Write (2,5700)
      MNWMIN = 1+2*MMAX*NMAX
      MNWMAX = 3*MMAX*NMAX
      Do 40 I = MNWMIN,MNWMAX
       REVEC(II) = DREAL(EVEC(I,1))
       AGEVEC = DIMAG(EVEC(I,1))
       If (ABS(AGEVEC).GT.1.0D-15) Then
        Write (2,5800) I,II,REVEC(II)
       Else
```

```
      Write (2,5900) I,II,REVEC(II)
      End If
      II = II+1
 40   Continue
c
c   DETERMINE W(X=A/2,Y)
c
      ASTEP = A/50.0
      BSTEP = B/50.0
      XCOORD = A/2.0
      YCOORD = 0.0
      Write (2,6000)
      Write (2,6100)
 50   WMODE = 0.0
      JJJ = 1
      Do 70 M = 1,MMAX
       Do 60 N = 1,NMAX
        WMODE = WMODE+REVEC(JJJ)*SIN(M*PI*XCOORD/A)*
     &   SIN(N*PI*YCOORD/B)
        JJJ = JJJ+1
 60    Continue
 70   Continue
      Write (2,6200) YCOORD,WMODE
      YCOORD = YCOORD+BSTEP
      If (YCOORD.GT.B) Then
       Go To 80
      Else
       Go To 50
      End If
c
 80   YCOORD = B/2.0
c
c   DETERMINE W(X, Y=B/2)
c
      XCOORD = 0.0
      Write (2,6300)
      Write (2,6400)
 90   WMODE = 0.0
      JJJ = 1
      Do 110 M = 1,MMAX
       Do 100 N = 1,NMAX
        WMODE = WMODE+REVEC(JJJ)*SIN(M*PI*XCOORD/A)*
     &   SIN(N*PI*YCOORD/B)
```

```
      JJJ = JJJ+1
 100  Continue
 110  Continue
      Write (2,6200) XCOORD,WMODE
      XCOORD = XCOORD+ASTEP
      If (XCOORD.GT.A) Then
       Go To 120
      Else
       Go To 90
      End If
C-----------------------------------------------------------
 120  Return
5000  Format (/,8X,I3,11X,'EIGENVALUE IS COMPLEX')
5100  Format (/,9X,I3,11X,'EIGENVALUE IS NEGATIVE')
5200  Format (/,9X,I3,11X,'EIGENVALUE IS INFINITE')
5300  Format (/,9X,I3,10X,D20.13,12X,D20.13)
5400  Format (/,8X,I3,10X,D20.13)
5500  Format (//,5X,'CRITICAL BUCKLING LOAD = ',1X,D20.13)
5600  Format (//,5X,'W EIGENVECTOR, CMN, FOR 1ST MODE')
5700  Format (//,5X,'M, N',10X,'CMN')
5800  Format (/,5X,I4,2X,I4,12X,D20.13,3X,'COMPLEX')
5900  Format (/,5X,I4,2X,I4,12X,D20.13)
6000  Format (//,5X,'DEFLECTION, W(X=A/2,Y)')
6100  Format (//,5X,'Y(IN.)',10X,'W(A/2,Y)(IN.)')
6200  Format (/,5X,F6.2,11X,E15.8)
6300  Format (//,5X,'DEFLECTION, W(X,Y=B/2)')
6400  Format (//,5X,'X(IN.)',10X,'W(X, B/2)(IN.)')
      End
```

Appendix C. Sample MACSYMA Batch File

MACSYMA (17, 26) was used to perform much of the long and complicated algebraic manipulations required for this thesis. The following is an example of the batch files used to perform the integration of the equations of motion. This sample generated the Galerkin equations for the clamped boundary condition, Case (1), as described in the Boundary Conditions section. (See Eqs (67) through (72))

Notice that this file reads another file called "force-matrix.mac." This file is not included here for brevity. It purpose was to develop the laminate constitutive relations to be substituted into the equations of motion for the resultant forces, moments, etc. (See Theory, Eqs 33 and 35).

```
$macsyma
loadfile("forcematrix.mac")$
DEPENDS([UO,VO,W,PSI],[X,Y])$
declare([m,n,p,q],integer)$
assume_pos:true$
uo:e[mn]*cos(m*%pi*x/a)*sin(n*%pi*y/b)$
vo:g[mn]*sin(m*%pi*x/a)*cos(n*%pi*y/b)$
w:c[mn]*sin(m*%pi*x/a)*sin(n*%pi*y/b)$
psi[x]:a[mn]*sin(m*%pi*x/a)*sin(n*%pi*y/b)$
psi[y]:b[mn]*sin(m*%pi*x/a)*sin(n*%pi*y/b)$
delu:cos(p*%pi*x/a)*sin(q*%pi*y/b)$
delv:sin(P*%pi*x/a)*cos(q*%pi*y/b)$
delw:sin(p*%pi*x/a)*sin(q*%pi*y/b)$
delpsix:sin(p*%pi*x/a)*sin(q*%pi*y/b)$
delpsiy:sin(p*%pi*x/a)*sin(q*%pi*y/b)$
E0[X]:DIFF(UO,X)$
E0[Y]:DIFF(VO,Y)+W/R$
G0[XY]:DIFF(UO,Y)+DIFF(VO,X)$
G0[YZ]:PSI[Y]+DIFF(W,Y)$
G0[XZ]:PSI[X]+DIFF(W,X)$
K0[X]:DIFF(PSI[X],X)$
K0[Y]:DIFF(PSI[Y],Y)$
K0[XY]:DIFF(PSI[X],Y)+DIFF(PSI[Y],X)+(DIFF(VO,X)-DIFF(UO,Y))
/(2*R)$
K1[Y]:-DIFF(PSI[Y],Y)/R$
K1[XY]:-DIFF(PSI[X],Y)/R$
K1[YZ]:3*K*(PSI[Y]+DIFF(W,Y))$
K1[XZ]:3*K*(PSI[X]+DIFF(W,X))$
K2[X]:K*(DIFF(PSI[X],X)+DIFF(W,X,2))$
K2[Y]:K*(DIFF(PSI[Y],Y)+DIFF(W,Y,2))$
K2[XY]:K*(DIFF(PSI[X],Y)+DIFF(PSI[Y],X)+2*DIFF(DIFF(W,X),Y))$
K3[Y]:-K*(DIFF(PSI[Y],Y)+DIFF(W,Y,2))/R$
K3[XY]:-K*(DIFF(PSI[X],Y)+DIFF(DIFF(W,X),Y))/R$
k:-4/(3*h^2)$
m:p;
n:q;
trigexpand:false$
writefile("clcase1.for");
uo1:(DIFF(EV(N[1]),X)+DIFF(EV(N[6]),Y)-DIFF(EV(M[6]),Y)/(2*R
))*delu$
uo1a:expand(ev(uo1))$
count:length(uo1a)$
uosum1:sum(integrate(integrate(part(uo1a,i),x,0,a),y,0,b),i,
1,count)$
ratsimp(uosum1,a[mn],b[mn],c[mn],e[mn],g[mn]);
vo1:(I2BP*OMS*PSI[Y]-I3BP*OMS*DIFF(W,Y)+DIFF(EV(N[2]),Y)+DIF
F(EV(N[6]),X)+DIFF(EV(M[6]),X)/(2*R))*delv$
vo1a:expand(ev(vo1))$
count:length(vo1a)$
```

```
vosuml:sum(integrate(integrate(part(vola,i),x,0,a),y,0,b),i,
1,count);$
ratsimp(%,a[mn],b[mn],c[mn],e[mn],g[mn]);
W1:(I5B*OMS*(DIFF(PSI[X],X)+DIFF(PSI[Y],Y))-K^2*I7*OMS*(DIFF
(W,X,2)+DIFF(W,Y,2))+I1*OMS*W+DIFF(EV(Q[1]),X)-K*(DIFF(EV(P[
1]),X,2)+DIFF(EV(P[2]),Y,2)+2*DIFF(DIFF(EV(P[6]),X),Y))+DIFF
(EV(Q[2]),Y)+3*K*(DIFF(EV(R[2]),Y)+DIFF(EV(R[1]),X))-(EV(N[2
])-K*(DIFF(EV(L[2]),Y,2)+DIFF(DIFF(EV(L[6]),X),Y)))/R+NBAR[1
]*DIFF(W,X,2)+2*NBAR[6]*DIFF(DIFF(W,X),Y)-NBAR[2]*(1/R-DIFF(
W,Y,2)))*delw$
wla:expand(ev(wl))$
count:length(wla)$
wsuml:sum(integrate(integrate(part(wla,i),x,0,a),y,0,b),i,1,
count)$
ratsimp(%,a[mn],b[mn],c[mn],e[mn],g[mn]);
psixl:(ib4*oms*psi[x]-ib5*oms*diff(w,x)+k*(diff(ev(p[1]),x)+
diff(ev(p[6]),y))+diff(ev(m[1]),x)+diff(ev(m[6]),y)-ev(q[1])
-3*k*ev(r[1])-(diff(ev(s[6]),y)+k*diff(ev(l[6]),y))/r)*delps
ix$
psixla:expand(ev(psixl))$
count:length(psixla)$
psixsuml:sum(integrate(integrate(part(psixla,i),x,0,a),y,0,b
),i,1,count)$
ratsimp(%,a[mn],b[mn],c[mn],e[mn],g[mn]);
psiyl:(ib4*oms*psi[y]-ib5*oms*diff(w,y)+k*(diff(ev(p[2]),y)+
diff(ev(p[6]),x))+diff(ev(m[2]),y)+diff(ev(m[6]),x)-ev(q[2])
-3*k*ev(r[2])-(diff(ev(s[2]),y)+k*diff(ev(l[2]),y))/r)*delps
iy$
psiyla:expand(ev(psiyl))$
count:length(psiyla)$
psiysuml:sum(integrate(integrate(part(psiyla,i),x,0,a),y,0,b
),i,1,count)$
ratsimp(%,a[mn],b[mn],c[mn],e[mn],g[mn]);
save("clcasel.mac",uosuml,vosuml,wsuml,psixsuml,psiysuml);
CLOSEFILE();
quit();
$exit
```

Appendix D. Galerkin Equations

This Appendix contains the Galerkin equations derived for the clamped and clamped-simple boundary conditions as described in Section II.

The Galerkin equations for Case (1) for the clamped boundary condition are as follows:

Equation (56) for u_o becomes:

$$
\begin{aligned}
& A_{mn}\cdot 0 + B_{mn}\cdot 0 \\
& - C_{mn}\{([(-32\pi^3E_{66} - 16\pi^3E_{12})a^2pq^2 - 16\pi^3E_{11}b^2p^3]R^2 + [(32\pi^3F_{66} \\
& \quad + 16\pi^3F_{12})a^2pq^2 - 12\pi A_{12}a^2b^2h^2p]R - 8\pi^3G_{66}a^2pq^2)/48a^2bh^2R^2\} \\
& - E_{mn}\{[(12\pi^2A_{66}a^3h^2q^2 + 12\pi^2A_{11}ab^2h^2p^2)R^2 - 12\pi^2B_{66}a^3h^2q^2R \\
& \quad + 3\pi^2D_{66}a^3h^2q^2]/48a^2bh^2R^2\} \\
& - G_{mn}\{[(12\pi^2A_{66} + 12\pi^2A_{12})a^2bh^2pqR^2 - 3\pi^2D_{66}a^2bh^2pq]/48a^2bh^2R^2\} \\
& \quad = 0
\end{aligned}
$$

Equation (57) for v_o becomes:

$$
\begin{aligned}
& A_{mn}\cdot 0 + B_{mn}\cdot 0 \\
& - C_{mn}\{([(-32\pi^3E_{66} - 16\pi^3E_{12})b^2p^2q - 16\pi^3E_{22}a^2q^3]R^2 + (16\pi^3F_{22}a^2q^3 \\
& \quad - 12\pi A_{22}a^2b^2h^2q)R + 8\pi^3G_{66}b^2p^2q)/48ab^2h^2R^2\} \\
& - E_{mn}\{[(12\pi^2A_{66} + 12\pi^2A_{12})ab^2h^2pqR^2 - 3\pi^2D_{66}ab^2h^2pq]/48ab^2h^2R^2\} \\
& - G_{mn}\{[(12\pi^2A_{22}a^2bh^2q^2 + 12\pi^2A_{66}b^3h^2p^2)R^2 + 12\pi^2B_{66}b^3h^2p^2R \\
& \quad + 3\pi^2D_{66}b^3h^2p^2]/48ab^2h^2R^2\} = \{\pi aq\overline{I}_3/4\}\omega^2C_{mn}
\end{aligned}
$$

Equation (58) for w becomes:

$$
\begin{aligned}
& A_{mn}\cdot 0 + B_{mn}\cdot 0 \\
& + C_{mn}\Big\{\Big[\big(-16\pi^4 H_{22}a^4q^4 + [(-64\pi^4 H_{66} - 32\pi^4 H_{12})a^2b^2p^2 - 9\pi^2 A_{44}a^4b^2h^4 \\
& \quad + 72\pi^2 D_{44}a^4b^2h^2 - 144\pi^2 F_{44}a^4b^2]q^2 - 16\pi^4 H_{11}b^4p^4 \\
& \quad + (-9\pi^2 A_{55}a^2b^4h^4 + 72\pi^2 D_{55}a^2b^4h^2 - 144\pi^2 F_{55}a^2b^4)p^2\big)R^2 \\
& \quad + \big(32\pi^4 I_{22}a^4q^4 + [(64\pi^4 I_{66} + 32\pi^4 I_{12})a^2b^2p^2 - 24\pi^2 E_{22}a^4b^2h^2]q^2 \\
& \quad - 24\pi^2 E_{12}a^2b^4h^2p^2\big)R - 16\pi^4 J_{22}a^4q^4 + (24\pi^2 F_{22}a^4b^2h^2 \\
& \quad - 16\pi^4 J_{66}a^2b^2p^2)q^2 - 9A_{22}a^4b^4h^4\Big]/36a^3b^3h^4R^2\Big\} \\
& + E_{mn}\Big\{\big([(24\pi^3 E_{66} + 12\pi^3 E_{12})a^3b^2h^2pq^2 + 12\pi^3 E_{11}ab^4h^2p^3]R^2 \\
& \quad + [(-24\pi^3 F_{66} - 12\pi^3 F_{12})a^3b^2h^2pq^2 + 9\pi A_{12}a^3b^4h^4p]R \\
& \quad + 6\pi^3 G_{66}a^3b^2h^2pq^2\big)/36a^3b^3h^4R^2\Big\} \\
& + G_{mn}\Big\{\big([12\pi^3 E_{22}a^4bh^2q^3 + (24\pi^3 E_{66} + 12\pi^3 E_{12})a^2b^3h^2p^2q]R^2 \\
& \quad + (9\pi A_{22}a^4b^3h^4q - 12\pi^3 F_{22}a^4bh^2q^3)R \\
& \quad - 6\pi^3 G_{66}a^2b^3h^2p^2q\big)/36a^3b^3h^4R^2\Big\} = -\Big\{[16\pi^2 I_7(a^2q^2 + b^2p^2) \\
& \quad + 9a^2b^2h^4 I_1]/36abh^4\Big\}\omega^2 C_{mn} + \{\pi^2bp^2/4a\}\overline{N}_1 C_{mn}
\end{aligned}
$$

Equation (59) for Ψ_x becomes:

$$- A_{mn} \{([(9\pi^2 D_{66} a^2 h^4 - 24\pi^2 F_{66} a^2 h^2 + 16\pi^2 H_{66} a^2) q^2 + (9\pi^2 D_{11} b^2 h^4$$

$$- 24\pi^2 F_{11} b^2 h^2 + 16\pi^2 H_{11} b^2) p^2 + 9 A_{55} a^2 b^2 h^4 - 72 D_{55} a^2 b^2 h^2$$

$$+ 144 F_{55} a^2 b^2] R^2 + (-18\pi^2 E_{66} a^2 h^4 + 48\pi^2 G_{66} a^2 h^2 - 32\pi^2 I_{66} a^2) q^2 R$$

$$+ (9\pi^2 F_{66} a^2 h^4 - 24\pi^2 H_{66} a^2 h^2 + 16\pi^2 J_{66} a^2) q^2) / 36 abh^4 R^2 \}$$

$$- B_{mn} \{([(9\pi^2 D_{26} a^2 h^4 - 24\pi^2 F_{26} a^2 h^2 + 16\pi^2 H_{26} a^2) q^2 + (9\pi^2 D_{16} b^2 h^4$$

$$- 24\pi^2 F_{16} b^2 h^2 + 16\pi^2 H_{16} b^2) p^2 + 9 A_{45} a^2 b^2 h^4 - 72\pi^2 D_{45} a^2 b^2 h^2$$

$$+ 144 F_{45} a^2 b^2] R^2 + (-18\pi^2 E_{26} a^2 h^4 + 48\pi^2 G_{26} a^2 h^2 - 32\pi^2 I_{26} a^2) q^2 R$$

$$+ (9\pi^2 F_{26} a^2 h^4 - 24\pi^2 H_{26} a^2 h^2 + 16\pi^2 J_{26} a^2) q^2) / 36 abh^4 R^2 \}$$

$$+ C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = 0$$

Equation (60) for Ψ_y becomes:

$$- A_{mn} \{([(9\pi^2 D_{26} a^2 h^4 - 24\pi^2 F_{26} a^2 h^2 + 16\pi^2 H_{26} a^2) q^2 + (9\pi^2 D_{16} b^2 h^4$$

$$- 24\pi^2 F_{16} b^2 h^2 + 16\pi^2 H_{16} b^2) p^2 + 9 A_{45} a^2 b^2 h^4 - 72 D_{45} a^2 b^2 h^2$$

$$+ 144 F_{45} a^2 b^2] R^2 + (-18\pi^2 E_{26} a^2 h^4 + 48\pi^2 G_{26} a^2 h^2 - 32\pi^2 I_{26} a^2) q^2 R$$

$$+ (9\pi^2 F_{26} a^2 h^4 - 24\pi^2 H_{26} a^2 h^2 + 16\pi^2 J_{26} a^2) q^2) / 36 abh^4 R^2 \}$$

$$- B_{mn} \{([(9\pi^2 D_{22} a^2 h^4 - 24\pi^2 F_{22} a^2 h^2 + 16\pi^2 H_{22} a^2) q^2 + (9\pi^2 D_{66} b^2 h^4$$

$$- 24\pi^2 F_{66} b^2 h^2 + 16\pi^2 H_{66} b^2) p^2 + 9 A_{44} a^2 b^2 h^4 - 72 D_{44} a^2 b^2 h^2$$

$$+ 144 F_{44} a^2 b^2] R^2 + (-18\pi^2 E_{22} a^2 h^4 + 48\pi^2 G_{22} a^2 h^2 - 32\pi^2 I_{22} a^2) q^2 R$$

$$+ (9\pi^2 F_{22} a^2 h^4 - 24\pi^2 H_{22} a^2 h^2 + 16\pi^2 J_{22} a^2) q^2) / 36 abh^4 R^2 \}$$

$$+ C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = 0$$

The Galerkin equations for Case (2) for the clamped boundary conditions are as follows:

Equation (56) for u_o becomes:

$$A_{mn}\{[(12\pi B_{16}h^2 - 16\pi E_{16})npqR + [(12\pi F_{16} - 9\pi D_{16}h^2)npq]/6Rh^2(q^2 - n^2)\}$$
$$+ B_{mn}\{([(6\pi B_{66} + 6\pi B_{12})h^2 - 8\pi E_{66} - 8\pi E_{12}]npqR + [(-3\pi D_{66} - 6\pi D_{12})h^2 + 4\pi F_{66} + 8\pi F_{12}]npq)/6h^2R(q^2 - n^2)\}$$
$$+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (57) for v_o becomes:

$$A_{mn}\{([(6\pi^2 B_{26}a^2h^2 - 8\pi^2 E_{26}a^2)nq^2 + (6\pi^2 B_{16}b^2h^2 - 8\pi^2 E_{16}b^2)np^2]R + (8\pi^2 F_{26}a^2 - 6\pi^2 D_{26}a^2h^2)nq^2 + (3\pi^2 D_{16}b^2h^2 - 4\pi^2 F_{16}b^2)np^2)/6\pi abh^2R(q^2 - n^2)\}$$
$$+ B_{mn}\{([(6\pi^2 B_{22}a^2h^2 - 8\pi^2 E_{22}a^2)nq^2 + (6\pi^2 B_{66}b^2h^2 - 8\pi^2 E_{66}b^2)np^2]R + (8\pi^2 F_{22}a^2 - 6\pi^2 D_{22}a^2h^2)nq^2 + (3\pi^2 D_{66}b^2h^2 - 4\pi^2 F_{66}b^2)np^2)/6\pi abh^2R(q^2 - n^2)\}$$
$$+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = \{abn\overline{I}_2'/\pi(q^2 - n^2)\}\omega^2 B_{mn}$$

Equation (58) for w becomes:

$$
\begin{aligned}
&- A_{mn} \{([(36\pi^2F_{16}b^2h^2 - 48\pi^2H_{16}b^2)np^2 + (12\pi^2F_{26}a^2h^2 \\
&\quad - 16\pi^2H_{26}a^2)n^3 + (-9A_{45}a^2b^2h^4 + 72D_{45}a^2b^2h^2 \\
&\quad - 144F_{45}a^2b^2)n]qR^2 + [(32\pi^2I_{16}b^2 - 24\pi^2G_{16}b^2h^2)np^2 \\
&\quad + (32\pi^2I_{26}a^2 - 24\pi^2G_{26}a^2h^2)n^3 + (9B_{26}a^2b^2h^4 \\
&\quad - 12E_{26}a^2b^2h^2)n]qR + [(12\pi^2H_{26}a^2h^2 - 16\pi^2J_{26}a^2)n^3 \\
&\quad + (12F_{26}a^2b^2h^2 - 9D_{26}a^2b^2h^4)n]q)/9abh^4R^2(q^2 - n^2)\} \\
&- B_{mn} \{[([(24\pi^2F_{66} + 12\pi^2F_{12})b^2h^2 + (-32\pi^2H_{66} - 16\pi^2H_{12})b^2]np^2 \\
&\quad + (12\pi^2F_{22}a^2h^2 - 16\pi^2H_{22}a^2)n^3 + (-9A_{44}a^2b^2h^4 + 72D_{44}a^2b^2h^2 \\
&\quad - 144F_{44}a^2b^2)n)qR^2 + ([(-12\pi^2G_{66} - 12\pi^2G_{12})b^2h^2 + (16\pi^2I_{66} \\
&\quad + 16\pi^2I_{12})b^2]np^2 + (32\pi^2I_{22}a^2 - 24\pi^2G_{22}a^2h^2)n^3 + (9B_{22}a^2b^2h^4 \\
&\quad - 12E_{22}a^2b^2h^2)n)qR + [(12\pi^2H_{22}a^2h^2 - 16\pi^2J_{22}a^2)n^3 \\
&\quad + (12F_{22}a^2b^2h^2 - 9D_{22}a^2b^2h^4)n]q]/9ab^2h^4R^2(q^2 - n^2)\} \\
&+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = -\{anq\overline{I}_5/(q^2 - n^2)\}\omega^2 B_{mn}
\end{aligned}
$$

Equation (59) for Ψ_x becomes;

$$
\begin{aligned}
& A_{mn} \cdot 0 + B_{mn} \cdot 0 \\
& - C_{mn} \Big\{ \big([(96\pi^2 H_{16} b^2 - 72\pi^2 F_{16} b^2 h^2) n p^2 + (32\pi^2 H_{26} a^2 \\
& \quad - 24\pi^2 F_{26} a^2 h^2) n^3 + (18 A_{45} a^2 b^2 h^4 - 144 D_{45} a^2 b^2 h^2 \\
& \quad + 288 F_{45} a^2 b^2) n] q R^2 + [(48\pi^2 G_{16} b^2 h^2 - 64\pi^2 I_{16} b^2) n p^2 \\
& \quad + (48\pi^2 G_{26} a^2 h^2 - 64\pi^2 I_{26} a^2) n^3 + (24 E_{26} a^2 b^2 h^2 \\
& \quad - 18 B_{26} a^2 b^2 h^4) n] q R + [(32\pi^2 J_{26} a^2 - 24\pi^2 H_{26} a^2 h^2) n^3 \\
& \quad + (18 D_{26} a^2 b^2 h^4 - 24 F_{26} a^2 b^2 h^2) n] q \big) / 18 a b^2 h^4 R^2 (q^2 - n^2) \Big\} \\
& - E_{mn} \Big\{ [(36\pi B_{16} a b^2 h^4 - 48\pi E_{16} a b^2 h^2) n p q R^2 + (36\pi F_{16} a b^2 h^2 \\
& \quad - 27\pi D_{16} a b^2 h^4) n p q R] / 18 a b^2 h^4 R^2 (q^2 - n^2) \Big\} \\
& - G_{mn} \Big\{ \big([(18\pi B_{16} b^3 h^4 - 24\pi E_{16} b^3 h^2) p^2 + (18\pi B_{26} a^2 b h^4 \\
& \quad - 24\pi E_{26} a^2 b h^2) n^2] q R^2 + [(9\pi D_{16} b^3 h^4 - 12\pi F_{16} b^3 h^2) p^2 \\
& \quad + (24\pi F_{26} a^2 b h^2 - 18\pi D_{26} a^2 b h^4) n^2] q R \big) / 18 a b^2 h^4 R^2 (q^2 - p^2) \Big\} \\
& = 0
\end{aligned}
$$

Equation (60) for Ψ_y becomes:

$$
\begin{aligned}
& A_{mn}\cdot 0 + B_{mn}\cdot 0 \\
& - C_{mn}\Big\{\Big[\Big([(-48\pi^2 F_{66} - 24\pi^2 F_{12})b^2h^2 + (64\pi^2 H_{66} + 32\pi^2 H_{12})b^2]np^2 \\
& \quad + (32\pi^2 H_{22}a^2 - 24\pi^2 F_{22}a^2h^2)n^3 + (18A_{44}a^2b^2h^4 - 144D_{44}a^2b^2h^2 \\
& \quad + 288F_{44}a^2b^2)n\Big)qR^2 - \Big([(24\pi^2 G_{66} + 24\pi^2 G_{12})b^2h^2 + (-32\pi^2 I_{66} \\
& \quad - 32\pi^2 I_{12})b^2]np^2 + (48\pi^2 G_{22}a^2h^2 - 64\pi^2 I_{22}a^2)n^3 + (24E_{22}a^2b^2h^2 \\
& \quad - 18B_{22}a^2b^2h^4)n\Big)qR - [(32\pi^2 J_{22}a^2 - 24\pi^2 H_{22}a^2h^2)n^3 \\
& \quad + (18D_{22}a^2b^2h^4 - 24F_{22}a^2b^2h^2)n]\,q\Big]/18ab^2h^4R^2(q^2 - n^2)\Big\} \\
& - E_{mn}\Big\{\Big([(18\pi B_{66} + 18\pi B_{12})ab^2h^4 + (-24\pi E_{66} - 24\pi E_{12})ab^2h^2]npqR^2 \\
& \quad + [(-9\pi D_{66} - 18\pi D_{12})ab^2h^4 + (12\pi F_{66} \\
& \quad + 24\pi F_{12})ab^2h^2]npqR\Big)/18ab^2h^4R^2(q^2 - n^2)\Big\} \\
& - G_{mn}\Big([(18\pi B_{66}b^3h^4 - 24\pi E_{66}b^3h^2)p^2 + (18\pi B_{22}a^2bh^4 \\
& \quad - 24\pi E_{22}a^2bh^2)n^2]qR^2 + [(9\pi D_{66}b^3h^4 - 12\pi F_{66}b^3h^2)p^2 \\
& \quad + (24\pi F_{22}a^2bh^2 - 18\pi D_{22}a^2bh^4)n^2]qR\Big)/18ab^2h^4R^2(q^2 - n^2)\Big\} \\
& \quad = \Big\{anq\overline{I}_5/(q^2 - n^2)\Big\}\omega^2 C_{mn}
\end{aligned}
$$

The Galerkin equations for Case (3) for the clamped boundary condition are as follows:

Equation (56) for u_o becomes:

$$
\begin{aligned}
&A_{mn}\Big\{\Big([(6\pi B_{66}a^2h^2 - 8\pi E_{66}a^2)mq^2 + (8\pi E_{11}b^2 - 6\pi B_{11}b^2h^2)mp^2 \\
&\quad + (12\pi B_{11}b^2h^2 - 16\pi E_{11}b^2)m^3]R^2 + (12\pi F_{66}a^2 - 9\pi D_{66}a^2h^2)mq^2R \\
&\quad + (3\pi E_{66}a^2h^2 - 4\pi G_{66}a^2)mq^2\Big)/6abh^2R^2(p^2 - m^2)\Big\} \\
&+ B_{mn}\Big\{\Big([(6\pi B_{26}a^2h^2 - 8\pi E_{26}a^2)mq^2 + (8\pi E_{16}b^2 - 6\pi B_{16}b^2h^2)mp^2 \\
&\quad + (12\pi B_{16}b^2h^2 - 16\pi E_{16}b^2)m^3]R^2 + (12\pi F_{26}a^2 - 9\pi D_{26}a^2h^2)mq^2R \\
&\quad + (3\pi E_{26}a^2h^2 - 4\pi G_{26}a^2)mq^2\Big)/6abh^2R^2(p^2 - m^2)\Big\} \\
&+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0
\end{aligned}
$$

Equation (57) for v_o becomes:

$$
\begin{aligned}
&A_{mn}\Big\{\Big([(6\pi B_{66} + 6\pi B_{12})h^2 - 8\pi E_{66} - 8\pi E_{12}]mpqR^2 + (4\pi F_{66} \\
&\quad - 3\pi D_{66}h^2)mpqR + (4\pi G_{66} - 3\pi E_{66}h^2)mpq\Big)/6h^2R^2(p^2 - m^2)\Big\} \\
&+ B_{mn}\Big\{[(12\pi B_{26}h^2 - 16\pi E_{26})mpqR^2 + (4\pi F_{26} - 3\pi D_{26}h^2)mpqR \\
&\quad + (4\pi G_{26} - 3\pi E_{26}h^2)mpq]/6h^2R^2(p^2 - m^2)\Big\} \\
&+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0
\end{aligned}
$$

Equation (58) for w becomes:

$$
\begin{aligned}
&- A_{mn} \Big\{\Big[\Big([(24\pi^2 F_{66} + 12\pi^2 F_{12}) a^2 h^2 + (-32\pi^2 H_{66} - 16\pi^2 H_{12}) a^2] mpq^2 \\
&\quad + [(12\pi^2 F_{11} b^2 h^2 - 16\pi^2 H_{11} b^2) m^3 + (-9A_{55} a^2 b^2 h^4 + 72 D_{55} a^2 b^2 h^2 \\
&\quad - 144 F_{55} a^2 b^2) m] p\Big) R^2 + \Big([(-36\pi^2 G_{66} - 12\pi^2 G_{12}) a^2 h^2 + (48\pi^2 I_{66} \\
&\quad + 16\pi^2 I_{12}) a^2] mpq^2 + (9B_{12} a^2 b^2 h^4 - 12 E_{12} a^2 b^2 h^2) mp\Big) R \\
&\quad + (12\pi^2 H_{66} a^2 h^2 - 16\pi^2 J_{66} a^2) mpq^2\Big] / 9a^2 b h^4 R^2 (p^2 - m^2)\Big\} \\
&- B_{mn} \Big\{\Big[\Big((36\pi^2 F_{26} a^2 h^2 - 48\pi^2 H_{26} a^2) mnq^2 + [(12\pi^2 F_{16} b^2 h^2 \\
&\quad - 16\pi^2 H_{16} b^2) m^3 + (-9A_{45} a^2 b^2 h^4 + 72 D_{45} a^2 b^2 h^2 \\
&\quad - 144 F_{45} a^2 b^2) m] p\Big) R^2 + [(64\pi^2 I_{26} a^2 - 48\pi^2 G_{26} a^2 h^2) mpq^2 \\
&\quad + (9B_{26} a^2 b^2 h^4 - 12 E_{26} a^2 b^2 h^2) mp] R + (12\pi^2 H_{26} \\
&\quad - 16\pi^2 J_{26} a^2) mpq^2\Big] / 9a^2 b h^4 R^2 (p^2 - m^2)\Big\} \\
&+ C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = -\Big\{bmp\overline{I}_5 / (p^2 - m^2)\Big\} \omega^2 A_{mn}
\end{aligned}
$$

Equation (59) for Ψ_x becomes:

$$
\begin{aligned}
& A_{mn}\cdot 0 + B_{mn}\cdot 0 \\
& - C_{mn}\Big\{\Big[\Big([(-48\pi^2 F_{66} - 24\pi^2 F_{12})a^2h^2 + (64\pi^2 H_{66} + 32\pi^2 H_{12})a^2]mpq^2 \\
& \quad + [(32\pi^2 H_{11}b^2 - 24\pi^2 F_{11}b^2h^2)m^3 + (18A_{55}a^2b^2h^4 - 144D_{55}a^2b^2h^2 \\
& \quad + 288F_{55}a^2b^2)m]p\Big)R^2 + \Big([(72\pi^2 G_{66} + 24\pi^2 G_{12})a^2h^2 + (-96\pi^2 I_{66} \\
& \quad - 32\pi^2 I_{12})a^2]mpq^2 + (24E_{12}a^2b^2h^2 - 18B_{12}a^2b^2h^4)mp\Big)R \\
& \quad + (32\pi^2 J_{66}a^2 - 24\pi^2 H_{66}a^2h^2)mpq^2\Big]/18a^2bh^4R^2(p^2 - m^2) \\
& - E_{mn}\Big\{\Big([(18\pi B_{66}a^3h^4 - 24\pi E_{66}a^3h^2)pq^2 + (18\pi B_{11}ab^2h^4 \\
& \quad - 24\pi E_{11}ab^2h^2)m^2p]R^2 + (36\pi F_{66}a^3h^2 - 27\pi D_{66}a^3h^4)pq^2R \\
& \quad + (9\pi E_{66}a^3h^4 - 12\pi G_{66}a^3h^2)pq^2\Big)/18a^2bh^4R^2(p^2 - m^2)\Big\} \\
& - G_{mn}\Big\{\Big([(18\pi B_{66} + 18\pi B_{12})a^2bh^4 + (-24\pi E_{66} - 24\pi E_{12})a^2bh^2]mpqR^2 \\
& \quad + (12\pi F_{66}a^2bh^2 - 9\pi D_{66}a^2bh^4)mpqR + (12\pi G_{66}a^2bh^2 \\
& \quad - 9\pi E_{66}a^2bh^4)mpq\Big)/18a^2bh^4R^2(p^2 - m^2)\Big\} \\
& \quad = \Big\{bmp\overline{I}_5/(p^2 - m^2)\Big\}\omega^2 C_{mn}
\end{aligned}
$$

Equation (60) for Ψ_y becomes:

$$A_{mn} \cdot 0 + B_{mn} \cdot 0$$

$$- C_{mn} \{[((96\pi^2 H_{26} a^2 - 72\pi^2 F_{26} a^2 h^2) mpq^2 + [(32\pi^2 H_{16} b^2$$

$$- 24\pi^2 F_{16} b^2 h^2) m^3 + (18 A_{45} a^2 b^2 h^4 - 144 D_{45} a^2 b^2 h^2$$

$$- 288 F_{45} a^2 b^2) m] p) R^2 + [(96\pi^2 G_{26} a^2 h^2 - 128\pi^2 I_{26} a^2) mpq^2$$

$$+ (24 E_{26} a^2 b^2 h^2 - 18 B_{26} a^2 b^2 h^4) mp] R + (32\pi^2 J_{26} a^2$$

$$- 24\pi^2 H_{26} a^2 h^2) mpq^2) / 18 a^2 b h^4 R^2 (p^2 - m^2)\}$$

$$- E_{mn} \{([(18\pi B_{26} a^3 h^4 - 24\pi E_{26} a^3 h^2) pq^2 + (18\pi B_{16} a b^2 h^4$$

$$- 24\pi E_{16} a b^2 h^2) m^2 p] R^2 + (36\pi F_{26} a^3 h^2 - 27\pi D_{26} a^3 h^4) pq^2 R$$

$$+ (9\pi E_{26} a^3 b^4 - 12\pi G_{26} a^3 h^2) pq^2) / 18 a^2 b h^4 R^2 (p^2 - m^2)\}$$

$$- G_{mn} \{[(36\pi B_{26} a^2 b h^4 - 48\pi E_{26} a^2 b h^2) mpqR^2 + (12\pi F_{26} a^2 b h^2$$

$$- 9\pi D_{26} a^2 b h^4) mpqR + (12\pi G_{26} a^2 b h^2$$

$$- 9\pi E_{26} a^2 b h^4) mpq] / 18 a^2 b h^4 R^2 (p^2 - m^2)\} = 0$$

The Galerkin equations for Case (4) for the clamped boundary conditions are as follows:

Equation (56) for u_o becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 + B_{mn} \cdot 0 \\
& + C_{mn} \Big\{ \big[(-32\pi^2 E_{16} b^2 mnp^2 - 16\pi^2 E_{26} a^2 mn^3 - 16\pi^2 E_{16} b^2 m^3 n) qR^2 \\
& \quad + [16\pi^2 F_{16} b^2 mnp^2 + 24\pi^2 F_{26} a^2 mn^3 + (8\pi^2 F_{16} b^2 m^3 \\
& \quad - 12 A_{26} a^2 b^2 h^2 m) n] qR + (6 B_{26} a^2 b^2 h^2 mn \\
& \quad - 8\pi^2 G_{26} a^2 mn^3) q \big] / 3\pi ab^2 h^2 (p^2 - m^2)(q^2 - n^2) \Big\} \\
& + E_{mn} \Big\{ [(12\pi A_{16} ab^2 h^2 np^2 + 12\pi A_{16} ab^2 h^2 m^2 n) qR^2 + (-6\pi B_{16} ab^2 b^2 np^2 \\
& \quad - 6\pi B_{16} ab^2 h^2 m^2 n) qR] / 3\pi ab^2 h^2 R^2 (p^2 - m^2)(q^2 - n^2) \Big\} \\
& + G_{mn} \Big\{ [(12\pi A_{16} b^3 h^2 mp^2 + 12\pi A_{26} a^2 bh^2 mn^2) qR^2 + (6\pi B_{16} b^3 h^2 mp^2 \\
& \quad - 6\pi B_{26} a^2 bh^2 mn^2) qR] / 3\pi ab^2 h^2 R^2 (p^2 - m^2)(q^2 - n^2) \Big\} = 0
\end{aligned}
$$

Equations (57) for v_o becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 + B_{mn} \cdot 0 \\
& + C_{mn} \Big\{ \big[[(-16\pi^2 E_{26} a^2 mn^3 - 16\pi^2 E_{16} b^2 m^3 n) p - 32\pi^2 E_{26} a^2 mnpq^2] R^2 \\
& \quad + \big(16\pi^2 F_{26} a^2 mnpq^2 + [8\pi^2 F_{26} a^2 mn^3 + (-8\pi^2 F_{16} b^2 m^3 \\
& \quad - 12 A_{26} a^2 b^2 h^2 m) n] p \big) R + (8\pi^2 G_{26} a^2 mn^3 \\
& \quad - 6 B_{26} a^2 b^2 h^2 mn) p \big] / 3\pi a^2 bh^2 R^2 (p^2 - m^2)(q^2 - n^2) \Big\} \\
& + E_{mn} \Big\{ [(12\pi A_{26} a^3 h^2 npq^2 + 12\pi A_{16} ab^2 h^2 m^2 np) R^2 + (6\pi B_{16} ab^2 h^2 m^2 np \\
& \quad - 6\pi B_{26} a^3 h^2 npq^2) R] / 3\pi a^2 bh^2 R^2 (p^2 - m^2)(q^2 - n^2) \Big\} \\
& + G_{mn} \Big\{ [(12\pi A_{26} a^2 bh^2 mpq^2 + 12\pi A_{26} a^2 bh^2 mn^2 p) R^2 + (6\pi B_{26} a^2 bh^2 mpq^2 \\
& \quad + 6\pi B_{26} a^2 bh^2 mn^2 p) R] / 3\pi a^2 bh^2 R^2 (p^2 - m^2)(q^2 - n^2) \Big\} = 0
\end{aligned}
$$

Equation (58) for w becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 + B_{mn} \cdot 0 \\
& - C_{mn} \Big\{\Big[\big([(-72\pi A_{45}a^2b^2h^4 + 576\pi D_{45}a^2b^2h^2 - 1152\pi F_{45}a^2b^2)m \\
& \quad - 256\pi^3 H_{16}b^2m^3]n - 256\pi^3 H_{26}a^2mn^3\big)pqR^2 + [384\pi^3 I_{26}a^2mn^3 \\
& \quad + (128\pi^3 I_{16}b^2m^3 - 192\pi E_{26}a^2b^2h^2m)n]pqR + (96\pi F_{26}a^2b^2h^2mn \\
& \quad - 128\pi^3 J_{26}a^2mn^3)pq\Big]/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& - E_{mn} \Big\{\big((48\pi^2 E_{26}a^3h^2n^3 + 144\pi^2 E_{16}ab^2h^2m^2n)pqR^2 + [(36A_{26}a^3b^2h^4 \\
& \quad - 72\pi^2 F_{16}ab^2h^2m^2)n - 72\pi^2 F_{26}a^3h^2n^3]pqR + (24\pi^2 G_{26}a^3h^2n^3 \\
& \quad - 18B_{26}a^3b^2h^4n)pq\big)/9\pi a^2b^2h^4R^2(p^2 - m^2)(q^2 - n^2)\Big\} \\
& - G_{mn} \; [(144\pi^2 E_{26}a^2bh^2mn^2 + 48\pi^2 E_{16}b^3h^2m^3)pqR^2 \\
& \quad + (-72\pi^2 F_{26}a^2bh^2mn^2 + 24\pi^2 F_{16}b^3h^2m^3 + 36A_{26}a^2b^3h^4m)pqR \\
& \quad + (18B_{26}a^2b^3h^4m - 24\pi^2 G_{26}a^2bh^2mn^2)pq]/9\pi a^2b^2h^4R^2(p^2 \\
& \quad - m^2)(q^2 - n^2)\} = 0
\end{aligned}
$$

Equation (59) for Ψ_x becomes:

$$
\begin{aligned}
& A_{mn} \Big\{[(72D_{16}h^4 - 192F_{16}h^2 + 128H_{16})mnpqR + (-72E_{16}h^4 \\
& \quad + 192G_{16}h^2 - 128I_{16})mnpq]/9h^4R(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + B_{mn} \Big\{\big([(36D_{66} + 36D_{12})h^4 + (-96F_{66} - 96F_{12})h^2 + 64H_{66} \\
& \quad + 64H_{12}]mnpqR + [(-36E_{66} - 36E_{12})h^4 + (96G_{66} + 96G_{12})h^2 \\
& \quad - 64I_{66} - 64I_{12}]mnpq\big)/9h^4R(p^2 - m^2)(q^2 - n^2)\Big\} \\
& + C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = 0
\end{aligned}
$$

Equation (60) for Ψ_y becomes:

$$A_{mn}\Big\{\big([(36D_{66} + 36D_{12})h^4 + (-96F_{66} - 96F_{12})h^2 + 64H_{66}$$

$$+ 64H_{12}]mnpqR + [(-36E_{66} - 36E_{12})h^4 + (96G_{66} + 96G_{12})h^2$$

$$- 64I_{66} - 64I_{12}]mnpq\big)/9h^4R(p^2 - m^2)(q^2 - n^2)\Big\}$$

$$+ B_{mn}\Big\{[(72D_{26}h^4 - 192F_{26}h^2 + 128H_{26})mnpqR + (-72E_{26}h^4$$

$$+ 192G_{26}h^2 - 128I_{26})mnpq]/9h^4R(p^2 - m^2)(q^2 - n^2)\Big\}$$

$$+ C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

The Galerkin equation for Case (1) for the clamped-simple boundary condition are as follows:

Equation (56) for u_o becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 \\
& - B_{mn} \{([(12\pi^2 B_{66} + 12\pi^2 B_{12}) a^2bh^2 + (-16\pi^2 E_{66} - 16\pi^2 E_{12}) a^2b] pqR^2 \\
& \quad + [(-6\pi^2 D_{66} - 12\pi^2 D_{12}) a^2bh^2 + (8\pi^2 F_{66} \\
& \quad + 16\pi^2 F_{12}) a^2b] pqR)/48a^2bh^2R\} \\
& - C_{mn} \{([(-32\pi^3 E_{66} - 16\pi^3 E_{12}) a^2pq^2 - 16\pi^3 E_{11} b^2p^3] R^2 + [(32\pi^3 F_{66} \\
& \quad + 16\pi^3 F_{12}) a^2pq^2 - 12\pi A_{12} a^2b^2h^2p] R - 8\pi^3 G_{66} a^2pq^2)/48a^2bh^2R^2\} \\
& - E_{mn} \{[(12\pi^2 A_{66} a^3h^2q^2 + 12\pi^2 A_{11} ab^2h^2p^2) R^2 - 12\pi^2 B_{66} a^3h^2q^2R \\
& \quad + 3\pi^2 D_{66} a^3h^2q^2] / 48a^2bh^2R^2\} \\
& - G_{mn} \{[(12\pi^2 A_{66} + 12\pi^2 A_{12}) a^2bh^2pqR^2 - 3\pi^2 D_{66} a^2bh^2pq] / 48a^2bh^2R^2\} \\
& \quad = 0
\end{aligned}
$$

Equation (57) for v_o becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 \\
& - B_{mn} \{([(12\pi^2 B_{22} a^2 b h^2 - 16\pi^2 E_{22} a^2 b) q^2 + (12\pi^2 B_{66} b^3 h^2 \\
& \quad - 16\pi^2 E_{66} b^3) p^2] R^2 + [(16\pi^2 F_{22} a^2 b - 12\pi^2 D_{22} a^2 b h^2) q^2 \\
& \quad + (6\pi^2 D_{66} b^3 h^2 - 8\pi^2 F_{66} b^3) p^2] R)/48 a b^2 h^2 R^2\} \\
& - C_{mn} \{([(-32\pi^3 E_{66} - 16\pi^3 E_{12}) b^2 p^2 q - 16\pi^3 E_{22} a^2 q^3] R^2 \\
& \quad + (16\pi^3 F_{22} a^2 q^3 - 12\pi A_{22} a^2 b^2 h^2 q) R \\
& \quad + 8\pi^3 G_{66} b^2 p^2 q)/48 a b^2 b^2 R^2\} \\
& - E_{mn} \{[(12\pi^2 A_{66} + 12\pi^2 A_{12}) a b^2 h^2 p q R^2 - 3\pi^2 D_{66} a b^2 h^2 p q]/48 a b^2 h^2 R^2\} \\
& - G_{mn} \{[(12\pi^2 A_{22} a^2 b h^2 q^2 + 12\pi^2 A_{66} b^3 h^2 p^2) R^2 + 12\pi^2 B_{66} b^3 h^2 p^2 R \\
& \quad + 3\pi^2 D_{66} b^3 h^2 p^2]/48 a b^2 h^2 R^2\} = -\{a b \overline{I}_2'/4\} \omega^2 B_{mn} + \{\pi a q \overline{I}_3'/4\} \omega^2 C_{mn}
\end{aligned}
$$

Equation (58) for w becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 \\
& + B_{mn} \Big\{\Big\{\Big[(12\pi^3 F_{22} a^4 b h^2 - 16\pi^3 H_{22} a^4 b) q^3 + \Big([(24\pi^3 F_{66} \\
& \quad + 12\pi^3 F_{12}) a^2 b^3 h^2 + (-32\pi^3 H_{66} - 16\pi^3 H_{12}) a^2 b^3] p^2 \\
& \quad - 9\pi A_{44} a^4 b^3 h^4 + 72\pi D_{44} a^4 b^3 h^2 - 144\pi F_{44} a^4 b^3\Big) q\Big] R^2 \\
& \quad + \Big[(32\pi^3 I_{22} a^4 b - 24\pi^3 G_{22} a^4 b h^2) q^3 + \Big([(-12\pi^3 G_{66} \\
& \quad - 12\pi^2 G_{12}) a^2 b^3 h^2 + (16\pi^3 I_{66} + 16\pi^3 I_{12}) a^2 b^3] p^2 + 9\pi B_{22} a^4 h^3 h^4 \\
& \quad - 12\pi E_{22} a^4 b^3 h^2\Big) q\Big] R + (12\pi^3 H_{22} a^4 b h^2 - 16\pi^3 J_{22} a^4 b) q^3 \\
& \quad + (12\pi F_{22} a^4 b^3 h^2 - 9\pi D_{22} a^4 b^3 h^4) q\Big\} / 36 a^3 b^3 h^4 R^2\Big\} \\
& + C_{mn} \Big\{\Big[\Big(-16\pi^4 H_{22} a^4 q^4 + [(-64\pi^4 H_{66} - 32\pi^4 H_{12}) a^2 b^2 p^2 - 9\pi^2 A_{44} a^4 b^2 h^4 \\
& \quad + 72\pi^2 D_{44} a^4 b^2 h^2 - 144\pi^2 F_{44} a^4 b^2] q^2 - 16\pi^4 H_{11} b^4 p^4 \\
& \quad + (-9\pi^2 A_{55} a^2 b^4 h^4 + 72\pi^2 D_{55} a^2 b^4 h^2 - 144 F_{55} a^2 b^4) p^2\Big) R^2 \\
& \quad + \Big(32\pi^4 I_{22} a^4 q^4 + [(64\pi^4 I_{66} + 32\pi^4 I_{12}) a^2 b^2 p^2 \\
& \quad - 24\pi^2 E_{22} a^4 b^2 h^2] q^2 - 24\pi^2 E_{12} a^2 b^4 h^2 p^2\Big) R - 16\pi^4 J_{22} a^4 q^4 \\
& \quad + (24\pi^2 F_{22} a^4 b^2 h^2 - 16\pi^4 J_{66} a^2 b^2 p^2) q^2 - 9 A_{22} a^4 b^4 h^4\Big] / 36 a^3 b^3 h^4 R^2\Big\} \\
& + E_{mn} \Big\{\Big([(24\pi^3 E_{66} + 12\pi^3 E_{12}) a^3 b^2 h^2 p q^2 + 12\pi^3 E_{11} a b^4 h^2 p^3] R^2 \\
& \quad + [(-24\pi^3 F_{66} - 12\pi^3 F_{12}) a^3 b^2 h^2 p q^2 + 9\pi A_{12} a^3 b^4 h^4 p] R \\
& \quad + 6\pi^3 G_{66} a^3 b^2 h^2 p q^2\Big) / 36 a^3 b^3 h^4 R^2\Big\} \\
& + G_{mn} \Big\{\Big([12\pi^3 E_{22} a^4 b h^2 q^3 + (24\pi^3 E_{66} + 12\pi^3 E_{12}) a^2 b^3 h^2 p^2 q] R^2 \\
& \quad + (9\pi A_{22} a^4 b^3 h^4 q - 12\pi^3 F_{22} a^4 b h^2 q^3) R \\
& \quad - 6\pi^3 G_{66} a^2 b^3 h^2 p^2 q\Big) / 36 a^3 b^3 h^4 R^2\Big\} = \{\pi a q \overline{I}_5 / 4\} \omega^2 B_{mn} \\
& \quad - \Big\{[16\pi^2 I_7 (a^2 q^2 + b^2 p^2) + 9 a^2 b^2 h^4 I_1] / 36 a b h^4\Big\} \omega^2 C_{mn} \\
& \quad + \{\pi b p^2 / 4a\} \overline{N}_1 C_{mn}
\end{aligned}
$$

Equation (59) for Ψ_x becomes:

$$- A_{mn} \{([(9\pi^2 D_{66} a^2 h^4 - 24\pi^2 F_{66} a^2 h^2 + 16\pi^2 H_{66} a^2) q^2 + (9\pi^2 D_{66} b^2 h^4$$
$$- 24\pi^2 b^2 h^2 + 16\pi^2 H_{11} b^2) p^2 + 9A_{55} a^2 b^2 h^4 - 72 D_{55} a^2 b^2 h^2$$
$$+ 144 F_{55} a^2 b^2] R^2 + (-18\pi^2 E_{66} a^2 h^4 + 48\pi^2 G_{66} a^2 h^2 - 32\pi^2 I_{66} a^2) q^2 R$$
$$+ (9\pi^2 F_{66} a^2 h^4 - 24\pi^2 H_{66} a^2 h^2 + 16\pi^2 J_{66} a^2) q^2)/36 abh^4 R^2$$
$$= -\{ab\overline{I}_4/4\}\omega^2 A_{mn}$$

Equation (60) for Ψ_y becomes:

$$A_{mn} \cdot 0$$
$$- B_{mn} \{([(18\pi^2 D_{22} a^2 bh^4 - 48\pi^2 F_{22} a^2 bh^2 + 32\pi^2 H_{22} a^2 b) q^2$$
$$+ (18\pi^2 D_{66} b^3 h^4 - 48\pi^2 F_{66} b^3 h^2 \; 32\pi^2 H_{66} b^3) p^2 + 18 A_{44} a^2 b^3 h^4$$
$$- 144 D_{44} a^2 b^3 h^2 + 288 F_{44} a^2 b^3] R^2 + (-36\pi^2 E_{22} a^2 bh^4$$
$$+ 96\pi^2 G_{22} a^2 bh^2 - 64\pi^2 I_{22} a^2 b) q^2 R + 18\pi^2 F_{22} a^2 bh^4 - 48\pi^2 H_{22} a^2 bh^2$$
$$+ 32\pi^2 J_{22} a^2 b) q^2)/72 ab^2 h^4 R^2\}$$
$$- C_{mn} \{\{[(32\pi^3 H_{22} a^2 - 24\pi^3 F_{22} a^2 h^2) q^3 + ([(-48\pi^3 F_{66} - 24\pi^3 F_{12}) b^2 h^2$$
$$+ (64\pi^3 H_{66} + 32\pi^3 H_{12}) b^3] p^2 + 18\pi A_{44} a^2 b^2 h^4 - 144\pi D_{44} a^2 b^2 h^2$$
$$+ 288\pi F_{44} a^2 b^2) q] R^2 + [(48\pi^3 G_{22} a^2 h^2 - 64\pi^3 I_{22} a^2) q^3 + ([(24\pi^3 G_{66}$$
$$+ 24\pi^3 G_{12}) b^2 h^2 + (-32\pi^3 I_{66} - 32\pi^2 I_{12}) b^2 [p^2 - 18\pi B_{22} a^2 b^2 h^4$$
$$+ 24\pi E_{22} a^2 b^2 h^2) q] R + 32\pi^3 J_{22} a^2 - 24\pi^3 H_{22} a^2 h^2) q^3$$
$$+ (18\pi D_{22} A^2 b^2 h^4 - 24\pi F_{22} a^2 b^2 h^2) q\}/72 ab^2 h^4 R^2\}$$
$$- E_{mn} \{([(18\pi^2 B_{66} + 18\pi^2 B_{12}) ab^2 h^4 + (24\pi^2 E_{66} - 24\pi^2 E_{12}) ab^2 h^2] pqR^2$$
$$+ [(-9\pi^2 D_{66} - 18\pi^2 D_{12}) ab^2 h^4 + (12\pi^2 F_{66}$$
$$+ 24\pi^2 F_{12}) ab^2 h^2] pqR)/72 ab^2 h^4 R^2\}$$

$$- G_{mn}\left\{\left(\left[(18\pi^2 B_{22}a^2bh^4 - 24\pi^2 E_{22}a^2bh^2)q^2 + (18\pi^2 B_{66}b^3h^4 - 24\pi^2 E_{66}b^3h^2)p^2\right]R^2 + \left[(24\pi^2 F_{22}a^2bh^2 - 18\pi^2 D_{22}a^2bh^4)q^2 + (9\pi^2 D_{66}b^3h^4 - 12\pi^2 F_{66}b^3h^2)p^2\right]R\right)/72ab^2h^4R^2\right\}$$

$$= -\left\{ab\overline{I}_4/4\right\}\omega^2 B_{mn} + \left\{\pi aq\overline{I}_5/4\right\}\omega^2 C_{mn}$$

The Galerkin equations for Case (2) for the clamped-simple boundary condition are as follows:

Equation (56) for u_o becomes:

$$A_{mn}\left\{\left[(12\pi B_{16}h^2 - 16\pi E_{16})npqR + (12\pi F_{16} - 9\pi D_{16}h^2)npq\right]/6h^2R(q^2 - n^2)\right\} + B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (57) for v_o becomes:

$$A_{mn}\left\{\left(\left[(6\pi B_{26}a^2h^2 - 8\pi E_{26}a^2)nq^2 + (6\pi B_{16}b^2h^2 - 8\pi E_{16}b^2)np^2\right]R + (8\pi F_{26}a^2 - 6\pi D_{26}a^2h^2)nq^2 + (3\pi D_{16}b^2h^2 - 4\pi F_{16}b^2)np^2\right)/6abh^2R(q^2 - n^2)\right\}$$

$$+ B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (58) for w becomes:

$$- A_{mn} \{([(36\pi^2 F_{16} b^2 h^2 - 48\pi^2 H_{16} b^2) np^2 + (12\pi^2 F_{26} a^2 h^2$$
$$- 16\pi^2 H_{26} a^2) n^3 + (-9A_{45} a^2 b^2 h^4 + 72D_{45} a^2 b^2 h^2$$
$$- 144F_{45} a^2 b^2) n] qR^2 + [(32\pi^2 I_{16} b^2 - 24\pi^2 G_{16} b^2 h^2) np^2$$
$$+ (32\pi^2 I_{26} a^2 - 24\pi^2 G_{26} a^2 h^2) n^3 + (9B_{26} a^2 b^2 h^4$$
$$- 12E_{26} a^2 b^2 h^2) n] qR + [(12\pi^2 H_{26} a^2 h^2 - 16\pi^2 J_{26} a^2) n^3$$
$$+ (12F_{26} a^2 b^2 h^2 - 9D_{26} a^2 b^2 h^4) n] q)/9ab^2 h^4 R^2 (q^2 - n^2)\}$$
$$+ B_{mn} \cdot 0 + C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = 0$$

Equation (59) for Ψ_x becomes:

$$A_{mn} \cdot 0$$
$$- B_{mn} \{([(18\pi^2 D_{16} b^3 h^4 - 48\pi^2 F_{16} b^3 h^2 + 32\pi^2 H_{16} b^3) p^2 + (18\pi^2 D_{26} a^2 bh^4$$
$$- 48\pi^2 F_{26} a^2 bh^2 + 32\pi^2 H_{26} a^2 b) n^2 + 18A_{45} a^2 b^3 h^4 - 144D_{45} a^2 b^3 h^2$$
$$+ 288F_{45} a^2 b^3] qR^2 + (-36\pi^2 E_{26} a^2 bh^4 + 96\pi^2 G_{26} a^2 bh^2$$
$$- 64\pi^2 I_{26} a^2 b) n^2 qR + (18\pi^2 F_{26} a^2 bh^4 - 48\pi^2 H_{26} a^2 bh^2$$
$$+ 32\pi^2 J_{26} a^2 b) n^2 q)/18\pi ab^2 h^4 R^2 (q^2 - n^2)\}$$
$$- C_{mn} \{([(96\pi^3 H_{16} b^2 - 72\pi^3 F_{16} b^2 h^2) np^2 + (32\pi^3 H_{26} a^2 - 24\pi^3 F_{26} a^2 h^2) n^3$$
$$+ (18\pi A_{45} a^2 b^2 h^4 - 144\pi D_{45} a^2 b^2 h^2 + 288\pi F_{45} a^2 b^2) n] qR^2$$
$$+ [(48\pi^3 G_{16} b^2 h^2 - 64\pi^3 I_{16} b^2) np^2 + (48\pi^3 G_{26} a^2 h^2 - 64\pi^3 I_{26} a^2) n^3$$
$$+ (24\pi E_{26} a^2 b^2 h^2 - 18\pi B_{26} a^2 b^2 h^4) n] qR + [(32\pi^3 J_{26} a^2$$
$$- 24\pi^3 H_{26} a^2 h^2) n^3 + (18\pi D_{26} a^2 b^2 h^4$$
$$- 24\pi F_{26} a^2 b^2 h^2) n] q)/18\pi ab^2 h^4 R^2 (q^2 - n^2)\}$$

$$- E_{mn} \{ [(36\pi^2 B_{16} ab^2 h^4 - 48\pi^2 E_{16} ab^2 h^2) npqR^2 + (36\pi^2 F_{16} ab^2 h^2$$

$$- 27\pi^2 D_{16} ab^2 h^4) npqR] / 18\pi ab^2 h^4 R^2 (q^2 - n^2) \}$$

$$- G_{mn} \{ ([(18\pi^2 B_{16} b^3 h^4 - 24\pi^2 E_{16} b^3 h^2) p^2 + (18\pi^2 B_{26} a^2 bh^4$$

$$- 24\pi^2 E_{26} a^2 bh^2) n^2] qR^2 + [(9\pi^2 D_{16} b^3 h^4 - 12\pi^2 F_{16} b^3 h^2) p^2$$

$$+ (24\pi^2 F_{26} a^2 bh^2 - 18\pi^2 D_{26} a^2 bh^4) n^2] qR) / 18\pi ab^2 h^4 R^2 (q^2 - n^2) \}$$

$$= 0$$

Equation (60) for Ψ_y becomes:

$$A_{mn} \{ ([(9\pi^2 D_{26} a^2 h^4 - 36\pi^2 F_{26} a^2 h^2 + 32\pi^2 H_{26} a^2) nq^2 + (9\pi^2 D_{16} b^2 h^4$$

$$- 24\pi^2 F_{16} b^2 h^2 + 16\pi^2 H_{16} b^2) np^2 + (12\pi^2 F_{26} a^2 h^2 - 16\pi^2 H_{26} a^2) n^3$$

$$+ (9 A_{45} a^2 b^2 h^4 - 72 D_{45} a^2 b^2 h^2 + 144 F_{45} a^2 b^2) n] R^2 + [(-18\pi^2 E_{26} a^2 h^4$$

$$+ 72\pi^2 G_{26} a^2 h^2 - 64\pi^2 I_{26} a^2) nq^2 + (32\pi^2 I_{26} a^2 - 24\pi^2 G_{26} a^2 h^2) n^3] R$$

$$+ (9\pi^2 F_{26} a^2 h^4 - 36\pi^2 H_{26} a^2 h^2 + 32\pi^2 J_{26} a^2) nq^2 + (12\pi^2 H_{26} a^2 h^2$$

$$- 16\pi^2 J_{26} a^2) n^3) / 9\pi abh^4 R^2 (q^2 - n^2) \}$$

$$+ B_{mn} \cdot 0 + C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = 0$$

The Galerkin equations for Case (3) for the clamped-simple boundary condition are as follows:

Equation (56) for u_o becomes:

$$A_{mn}\{([6\pi B_{66}a^2h^2 - 8\pi E_{66}a^2)mq^2 + (6\pi B_{11}b^2h^2 - 8\pi E_{11}b^2)mp^2]R^2 + (12\pi F_{66}a^2 - 9\pi D_{66}a^2h^2)mq^2R + (3\pi E_{66}a^2h^2 - 4\pi G_{66}a^2)mq^2)/6abh^2R^2(p^2 - m^2)\}$$
$$+ B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (57) for v_o becomes:

$$A_{mn}\{([(6\pi B_{66} + 6\pi B_{12})h^2 - 8\pi E_{66} - 8\pi E_{12}]mpqR^2 + (4\pi F_{66} - 3\pi D_{66}h^2)mpqR + (4\pi G_{66} - 3\pi E_{66}h^2)mpq)/6h^2R^2(p^2 - m^2)\}$$
$$+ B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (58) for w becomes:

$$
\begin{aligned}
&- A_{mn} \Big\{\Big[\Big([(24\pi^2 F_{66} + 12\pi^2 F_{12}) a^2 h^2 + (-32\pi^2 H_{66} - 16\pi^2 H_{12}) a^2] mpq^2 \\
&\quad + [(12\pi^2 F_{11} b^2 h^2 - 16\pi^2 H_{11} b^2) m^3 + (-9A_{55} a^2 b^2 h^4 + 72 D_{55} a^2 b^2 h^2 \\
&\quad - 144 F_{55} a^2 b^2) m] p\Big) R^2 + \Big([(-36\pi^2 G_{66} - 12\pi^2 G_{12}) a^2 h^2 + (48\pi^2 I_{66} \\
&\quad + 16\pi^2 I_{12}) a^2] mpq^2 + (9 B_{12} a^2 b^2 h^4 - 12 E_{12} a^2 b^2 h^2) mp\Big) R \\
&\quad + (12\pi^2 H_{66} a^2 h^2 - 16\pi^2 H_{66} a^2) mpq^2\Big] / 9 a^2 b h^4 R^2 (p^2 - m^2)\Big\} \\
&+ B_{mn} \cdot 0 + C_{mn} \cdot 0 + E_{mn} \cdot 0 + G_{mn} \cdot 0 = -\{bmp\overline{I}_5 / (p^2 - m^2)\} \omega^2 A_{mn}
\end{aligned}
$$

Equation (59) for Ψ_x becomes:

$$
\begin{aligned}
&A_{mn} \cdot 0 \\
&- B_{mn} \Big\{\Big([(18\pi D_{66} + 18\pi D_{12}) a^2 b h^4 + (-48\pi F_{66} - 48\pi F_{12}) a^2 b h^2 \\
&\quad + (32\pi H_{66} + 32\pi H_{12}) a^2 b] mpqR^2 + [(-18\pi E_{66} - 18\pi E_{12}) a^2 b h^4 \\
&\quad + (48\pi G_{66} + 48\pi G_{12}) a^2 b h^2 + (-32\pi I_{66} \\
&\quad - 32\pi I_{12}) a^2 b] mpqR\Big) / 18 a^2 b h^4 R^2 (p^2 - m^2)\Big\} \\
&- C_{mn} \Big\{\Big[\Big([(-48\pi^2 F_{66} - 24\pi^2 F_{12}) a^2 h^2 + (64\pi^2 H_{66} + 32\pi^2 H_{12}) a^2] mpq^2 \\
&\quad + [(32\pi^2 H_{11} b^2 - 24\pi^2 F_{11} b^2 h^2) m^3 + (18 A_{55} a^2 b^2 h^4 - 144 D_{55} a^2 b^2 h^2 \\
&\quad + 288 F_{55} a^2 b^2) m] p\Big) R^2 + \Big([(72\pi^2 G_{66} + 24\pi^2 G_{12}) a^2 h^2 + (-96\pi^2 I_{66} \\
&\quad - 32\pi^2 I_{12}) a^2] mpq^2 + (24 E_{12} a^2 b^2 h^2 - 18 B_{12} a^2 b^2 h^4) mp\Big) R \\
&\quad + (32\pi^2 J_{66} a^2 - 24\pi^2 H_{66} a^2 h^2) mpq^2\Big] / 18 a^2 b h^4 R^2 (p^2 - m^2)\Big\} \\
&- E_{mn} \Big\{\Big([(18\pi B_{66} a^3 h^4 - 24\pi E_{66} a^3 h^2) pq^2 + (18\pi B_{11} a b^2 h^4 \\
&\quad - 24\pi E_{11} a b^2 h^2) m^2 p] R^2 + (36\pi F_{66} a^3 h^2 - 27\pi D_{66} a^3 h^4) pq^2 R \\
&\quad + (9\pi E_{66} a^3 h^4 - 12\pi G_{66} a^3 h^2) pq^2\Big) / 18 a^2 b h^4 R^2 (p^2 - m^2)\Big\}
\end{aligned}
$$

$$- G_{mn}\left\{\left(\left[(18\pi B_{66} + 18\pi B_{12})a^2bh^4 + (-24\pi E_{66} - 24\pi E_{12})a^2bh^2\right]mpqR^2\right.\right.$$

$$+ (12\pi F_{66}a^2bh^2 - 9\pi D_{66}a^2bh^4)mpqR + (12\pi G_{66}a^2bh^2$$

$$\left.\left. - 9\pi E_{66}a^2bh^4)mpq\right)/18a^2bh^4R^2(p^2-m^2)\right\} = \left\{bmp\overline{I}_5/(p^2-m^2)\right\}\omega^2 C_{mn}$$

Equation (60) for Ψ_y becomes:

$$A_{mn}\left\{\left(\left[(9\pi D_{66} + 9\pi D_{12})h^4 + (-24\pi F_{66} - 24\pi F_{12})h^2 + 16\pi H_{66}\right.\right.\right.$$

$$+ 16\pi H_{12}]mpqR + [(-9\pi E_{66} - 9\pi E_{12})h^4 + (24\pi G_{66} + 24\pi G_{12})h^2$$

$$\left.\left. - 16\pi I_{66} - 16\pi I_{12}]mpq\right)/9h^4R(p^2 - m^2)\right\}$$

$$+ B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

The Galerkin equations for Case (4) for the clamped-simple boundary condition are as follows:

Equation (56) for u_o becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 \\
& + B_{mn}\Big\{\big([(12\pi B_{16}b^3h^2 - 16\pi E_{16}b^3)mp^2 + (12\pi B_{26}a^2bh^2 \\
& \quad - 16\pi E_{26}a^2b)mn^2]qR^2 + (24\pi F_{26}a^2b - 18\pi D_{26}a^2bh^2)mn^2qR \\
& \quad + (6\pi E_{26}a^2bh^2 - 8\pi G_{26}a^2b)mn^2q\big)/3\pi ab^2h^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
& + C_{mn}\Big\{\big((-32\pi^2E_{16}b^2mnp^2 - 16\pi^2E_{26}a^2mn^3 - 16\pi^2E_{16}b^2m^3n)qR^2 \\
& \quad + [16\pi^2F_{16}b^2mnp^2 + 24\pi^2F_{26}a^2mn^3 + (8\pi^2F_{16}b^2m^3 \\
& \quad - 12A_{26}a^2b^2h^2m)n]qR + (6B_{26}a^2b^2h^2mn \\
& \quad - 8\pi^2G_{26}a^2mn^3)q\big)/3\pi ab^2h^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
& + E_{mn}\Big\{[(12\pi A_{16}ab^2h^2np^2 + 12\pi A_{16}ab^2h^2m^2n)qR + (-6\pi B_{16}ab^2h^2np^2 \\
& \quad - 6\pi B_{16}ab^2h^2m^2n)qR]/3\pi ab^2h^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
& + G_{mn}\Big\{[(12\pi A_{16}b^3h^2mp^2 + 12\pi A_{26}a^2bh^2mn^2)qR^2 + (6\pi B_{16}b^3h^2mp^2 \\
& \quad - 6\pi B_{26}a^2bh^2mn^2)qR]/3\pi ab^2h^2R^2(p^2-m^2)(q^2-n^2)\Big\} = 0
\end{aligned}
$$

Equation (57) for v_o becomes:

$$
\begin{aligned}
&A_{mn} \cdot 0 \\
&+ B_{mn} \Big\{\Big([(12\pi B_{26}a^2bh^2 - 16\pi E_{26}a^2b)mpq^2 + (12\pi B_{26}a^2bh^2 \\
&\quad - 16\pi E_{26}a^2b)mn^2p]R^2 + (8\pi F_{26}a^2b - 6\pi D_{26}a^2bh^2)mn^2pR \\
&\quad + (8\pi G_{26}a^2b - 6\pi E_{26}a^2bh^2)mn^2p\Big)/3\pi a^2bh^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
&+ C_{mn} \Big\{\Big[[(-16\pi^2E_{26}a^2mn^3 - 16\pi^2E_{16}b^2m^3n)p - 32\pi^2E_{26}a^2mnpq^2]R^2 \\
&\quad + \Big(16\pi^2F_{26}a^2mnpq^2 + [8\pi^2F_{26}a^2mn^3 + (-8\pi^2F_{16}b^2m^3 \\
&\quad - 12A_{26}a^2b^2h^2m)n]p\Big)R + (8\pi^2G_{26}a^2mn^3 \\
&\quad - 6B_{26}a^2b^2h^2mn)p]/3\pi a^2bh^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
&+ E_{mn} \Big\{[(12\pi A_{26}a^3h^2npq^2 + 12\pi A_{16}ab^2h^2m^2np)R^2 + (6\pi B_{16}ab^2h^2m^2np \\
&\quad - 6\pi B_{26}a^3h^2npq^2)R]/3\pi a^2bh^2R^2(p^2-m^2)(q^2-n^2)\Big\} \\
&+ G_{mn} \Big\{[(12\pi A_{26}a^2bh^2mpq^2 + 12\pi A_{26}a^2bh^2mn^2p)R^2 + (6\pi B_{26}a^2bh^2mpq^2 \\
&\quad + 6\pi B_{26}a^2bh^2mn^2p)R]/3\pi a^2bh^2R^2(p^2-m^2)(q^2-n^2)\Big\} = 0
\end{aligned}
$$

Equation (58) for w becomes:

$$
\begin{aligned}
&A_{mn} \cdot 0 \\
&- B_{mn} \Big\{\Big([(144\pi^2F_{26}a^2bh^2 - 192\pi^2H_{26}a^2b)mn^2 + (48\pi^2F_{16}b^3h^2 \\
&\quad - 64\pi^2H_{16}b^3)m^3 + (-36A_{45}a^2b^3h^4 + 288D_{45}a^2b^3h^2 \\
&\quad - 576F_{45}a^2b^3)m]pqR^2 + [(256\pi^2I_{26}a^2b - 192\pi^2G_{26}a^2bh^2)mn^2 \\
&\quad + (36B_{26}a^2b^3h^4 - 48E_{26}a^2b^3h^2)m]pqR + (48\pi^2H_{26}a^2bh^2 \\
&\quad - 64\pi^2J_{26}a^2b)mn^2pq\Big)/9\pi a^2b^2h^4R^2(p^2-m^2)(q^2-n^2)\Big\}
\end{aligned}
$$

$$- C_{mn}\Big\{\Big[\Big(\big[(-72\pi A_{45}a^2b^2h^4 + 576\pi D_{45}a^2b^2h^2 - 1152\pi F_{45}a^2b^2)m$$

$$- 256\pi^3 H_{16}b^2m^3\big]n - 256\pi^3 H_{26}a^2mn^3\Big)pqR^2 + [384\pi^3 I_{26}a^2mn^3$$

$$+ (128\pi^3 I_{16}b^2m^3 - 192\pi E_{26}a^2b^2h^2m)n]pqR + (96\pi F_{26}a^2b^2h^2mn$$

$$- 128\pi^3 H_{26}a^2mn^3)pq\Big]/9\pi a^2b^2h^4R^2(p^2-m^2)(q^2-n^2)\Big\}$$

$$- E_{mn}\Big\{\Big((48\pi^2 E_{26}a^3h^2n^3 + 144\pi^2 E_{16}ab^2h^2m^2n)pqR^2 + [(36A_{26}a^3b^2h^4$$

$$- 72\pi^2 F_{16}ab^2h^2m^2)n - 72\pi^2 F_{26}a^3h^2n^3]pqR + (24\pi^2 G_{26}a^3h^2n^3$$

$$- 18B_{26}a^3b^2h^4n)pq\Big)/9\pi a^2b^2h^4R^2(p^2-m^2)(q^2-n^2)\Big\}$$

$$- G_{mn}\Big\{[(144\pi^2 E_{26}a^2bh^2mn^2 + 48\pi^2 E_{16}b^3h^2m^3)pqR^2$$

$$+ (-72\pi^2 F_{26}a^2bh^2mn^2 + 24\pi^2 F_{16}b^3h^2m^3 + 36A_{26}a^2b^3h^4m)pqR$$

$$+ (18B_{26}a^2b^3h^4m - 24\pi^2 G_{26}a^2bh^2mn^2)pq]/9\pi a^2b^2h^4R^2(p^2$$

$$-m^2)(q^2-n^2)\Big\} = 0$$

Equation (59) for Ψ_x becomes:

$$A_{mn}\Big\{[(72D_{16}h^4 - 192F_{16}h^2 + 128H_{16})mnpqR + (-72E_{16}h^4$$

$$+ 192G_{16}h^2 - 128I_{16})mnpq]/9h^4R(p^2-m^2)(q^2-n^2)\Big\}$$

$$+ B_{mn}\cdot 0 + C_{mn}\cdot 0 + E_{mn}\cdot 0 + G_{mn}\cdot 0 = 0$$

Equation (60) for Ψ_y becomes:

$$
\begin{aligned}
& A_{mn} \cdot 0 \\
& + B_{mn} \{([(36\pi D_{26} a^2 b h^4 - 144\pi F_{26} a^2 b h^2 + 128\pi H_{26} a^2 b) mpq^2 \\
& \quad + (36\pi D_{26} a^2 b h^4 - 48\pi F_{26} a^2 b h^2) mn^2 p] R^2 + [(36\pi E_{26} a^2 b h^4 \\
& \quad + 144\pi G_{26} a^2 b h^2 - 128\pi I_{26} a^2 b) mpq^2 + (48\pi G_{26} a^2 b h^2 \\
& \quad - 36\pi E_{26} a^2 b h^4) mn^2 p] R)/9\pi a^2 b h^4 R^2 (p^2 - m^2)(q^2 - n^2)\} \\
& + C_{mn} \{\{[(256\pi^2 H_{26} a^2 - 96\pi^2 F_{26} a^2 h^2) mnpq^2 + ((-48\pi^2 F_{26} a^2 h^2 \\
& \quad - 64\pi^2 H_{26} a^2) mn^3 + [(64\pi^2 H_{16} b^2 - 48\pi^2 F_{16} b^2 h^2) m^3 \\
& \quad + (36 A_{45} a^2 b^2 h^4 - 288 D_{45} a^2 b^2 h^2 + 576 F_{45} a^2 b^2) m] n) p] R^2 \\
& \quad + ((144\pi^2 G_{26} a^2 h^2 - 384\pi^2 I_{26} a^2) mnpq^2 + [(48\pi^2 G_{26} a^2 h^2 \\
& \quad + 128\pi^2 I_{26} a^2) mn^3 + (48 E_{26} a^2 b^2 h^2 - 36 B_{26} a^2 b^2 h^4) mn] p) R \\
& \quad + (128\pi^2 J_{26} a^2 - 48\pi^2 H_{26} a^2 h^2) mnpq^2 \\
& \quad - 64\pi^2 J_{26} a^2 mn^3 p\}/9\pi a^2 b h^4 R^2 (p^2 - m^2)(q^2 - n^2)\} \\
& + E_{mn} \{[((36\pi B_{26} a^3 h^4 - 94\pi E_{26} a^3 h^2) npq^2 + [48\pi E_{26} a^3 h^2 n^3 \\
& \quad + (36\pi B_{16} a b^2 h^4 - 48\pi E_{16} a b^2 h^2) m^2 n] p) R^2 + [(144\pi F_{26} a^3 h^2 \\
& \quad - 54\pi D_{26} a^3 h^4) npq^2 - 72\pi F_{26} a^3 h^2 n^3 p] R + (18\pi E_{26} a^3 h^4 \\
& \quad - 48\pi G_{26} a^3 h^2) npq^2 + 24\pi G_{26} a^3 h^2 n^3 p]/9\pi a^2 b h^4 R^2 (p^2 - m^2)(q^2 - n^2)\} \\
& + G_{mn} \{([(36\pi B_{26} a^2 b h^4 - 96\pi E_{26} a^2 b h^2) mpq^2 + 36\pi B_{26} a^2 b h^4 mn^2 p] R^2 \\
& \quad + [(48\pi F_{26} a^2 b h^2 - 18\pi D_{26} a^2 b h^4) mpq^2 - 24\pi F_{26} a^2 b h^2 mn^2 p] R \\
& \quad + (48\pi G_{26} a^2 b h^2 - 18\pi E_{26} a^2 b h^4) mpq^2 \\
& \quad - 24\pi G_{26} a^2 b h^2 mn^2 p)/9\pi a^2 b h^4 R^2 (p^2 - m^2)(q^2 - n^2)\} = 0
\end{aligned}
$$

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An analytical study was performed to determine the critical buckling loads and natural frequencies for composite cylindrical shells, including transverse shear and constant through the thickness direct strain. A linearized form of Sanders' shell equations are derived including a parabolic shear strain distribution. Higher order laminate constitutive relations are developed. Hamilton's Principle is applied to derive five equations of motion and boundary conditions, which are solved using the Galerkin technique. Various ply layups are investigated with three boundary conditions, siimply supported, clamped, and a combination of simple and clamped. Symmetric and nonsymmetric laminates were investigated. Curvature is shown to have an important effect on the buckling loads, due to membrane and bending coupling. Curved panels have significantly higher buckling loads than flat plates for all laminates studied. Frequencies were not as affected. Comparisons of various laminates show results are greatly dependent on geometry, curvature, and boundary conditions. Most of the results for the nonsymmetric laminates unexpectedly indicated those to be stiffer than the symmetric.

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